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and other
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Matrix Inversion by a Monte Carlo Method[1]

The following unusual method of inverting a class of matrices was devised by J. VON NEUMANN and S. M. ULAM. Since it appears not to be in print, an exposition may be of interest to readers of *MTAC*. The method is remarkable in that it can be used to invert a class of n-th order matrices (see final paragraph) with only n^2 arithmetic operations in addition to the scanning and discriminating required to play the solitaire game. The method therefore appears best suited to a human computer with a table of random digits and no calculating machine. Moreover, the method lends itself fairly well to obtaining a single element of the inverse matrix without determining the rest of the matrix. The term "Monte Carlo" refers to mathematical sampling procedures used to approximate a theoretical distribution [see *MTAC*, v. 3, p. 546].

Let B be a matrix of order n whose inverse is desired, and let $A = I - B$, where I is the unit matrix. For any matrix M, let $\lambda_r(M)$ denote the r-th proper value of M, and let M_{ij} denote the element of M in the i-th row and j-th column. The present method presupposes that

$$\max_r |1 - \lambda_r(B)| = \max_r |\lambda_r(A)| < 1. \tag{1}$$

When (1) holds, it is known that

$$(B^{-1})_{ij} = ([I - A]^{-1})_{ij} = \sum_{k=0}^{\infty} (A^k)_{ij}. \tag{2}$$

The Monte Carlo method to compute $(B^{-1})_{ij}$ is to play a solitaire game $\mathbf{G}_{ij}$ whose expected payment is $(B^{-1})_{ij}$. According to a result of KOLMOGOROFF[2] on the strong law of large numbers, if one plays such a game repeatedly, the average payment for N successive plays will converge to $(B^{-1})_{ij}$ as $N \to \infty$, for almost all sequences of plays. The rules of the game will be expressed in terms of balls in urns, but a computer would probably use a table of random digits.

For $1 \leqq i,j \leqq n$ pick probabilities $p_{ij} \geqq 0$ and corresponding "value factors" v_{ij}, subject only to the conditions that $p_{ij}v_{ij} = a_{ij}$ and $\sum_{j=1}^{n} p_{ij} < 1$. Let the "stop probabilities" p_i be defined by the relations $p_i = 1 - \sum_{j=1}^{n} p_{ij}$. Take n urns. In the i-th urn $U_i(i = 1, 2, \cdots, n)$ put an assortment of $n + 1$ different types of balls. Each ball of the j-th type is marked "j," and will be drawn from U_i with probability $p_{ij}(j = 1, 2, \cdots, n)$. The $(n + 1)$-th type of ball is marked "STOP," and will be drawn from U_i with probability p_i.

For the game $\mathbf{G}_{ij}$ now to be defined, the value of a play is a random variable G_{ij} whose expectation will be proved to be $(B^{-1})_{ij}$. First draw a ball from U_i (all drawings are with replacement). If it is a STOP ball, the payment G_{ij} is 0 (if $i \neq j$) or p_j^{-1} (if $i = j$). Otherwise the ball must carry a mark $i_1(1 \leqq i_1 \leqq n)$, and one is next to draw a ball from U_{i_1}, which in turn tells one whether to stop or draw again. One follows this treasure hunt from urn to urn until a STOP ball is first drawn—say from U_h on the k-th drawing. If $h \neq j$, the payment G_{ij} is 0. If $h = j$, suppose one has

arrived at U_j via a route $i = i_0 \to i_1 \to i_2 \to \cdots \to i_{k-1} \to j$, whose abbreviation is ρ. Then the payment G_{ij} is defined to be

$$V_\rho p_j^{-1} = v_{i_0 i_1} v_{i_1 i_2} \cdots v_{i_{k-1} j} p_j^{-1}.$$

THEOREM 1. *If* (1) *holds,* G_{ij} *has the expectation* $(B^{-1})_{ij}$.

Proof. The probability of following the route ρ and then drawing a STOP ball is

$$P_\rho p_j = p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{k-1} j} p_j.$$

The expected payment in the game $\mathbf{G}_{ij}$ is

$$E(G_{ij}) = \sum_\rho (P_\rho p_j)(V_\rho p_j^{-1}) = \sum_\rho P_\rho V_\rho,$$

where the sum is extended over all routes ρ from i to j. Since $p_{ij} v_{ij} = a_{ij}$,

$$\begin{aligned} E(G_{ij}) &= \delta_{ij} + \sum_{k=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &= I_{ij} + \sum_{k=1}^{\infty} (A^k)_{ij}, \end{aligned}$$

where δ_{ij} is the KRONECKER symbol. By (2) the theorem is proved.

Let us calculate the variance σ_{ij}^2 of the payment G_{ij} about its expected value. Let R be the matrix $(a_{ij} v_{ij}) = (p_{ij} v_{ij}^2)$. If $\max_i |\lambda_i(R)| \geqq 1$, the proof below can be modified[3] to show that $\sigma_{ij} = \infty$. Assume $\max_i |\lambda_i(R)| < 1$, and define $(I - R)^{-1} = T$.

THEOREM 2. *If* $\max_i |\lambda_i(R)| < 1$,

$$\sigma_{ij}^2 = T_{ij} p_j^{-1} - (B^{-1})_{ij}^2.$$

Proof. Using the above notation

$$\begin{aligned} \sigma_{ij}^2 &= E\{G_{ij} - (B^{-1})_{ij}\}^2 = E(G_{ij}^2) - (B^{-1})_{ij}^2 \\ &= \sum_\rho (P_\rho p_j)(V_\rho p_j^{-1})^2 - (B^{-1})_{ij}^2 \\ &= p_j^{-1} \sum_\rho P_\rho V_\rho^2 - (B^{-1})_{ij}^2. \end{aligned}$$

If $\max_i |\lambda_i(R)| < 1$, the end of the proof of Theorem 1 can be modified to show that the last sum is T_{ij}. This proves Theorem 2.

One can of course compute one whole row of B^{-1} at once by playing the games $\mathbf{G}_{i\alpha} (\alpha = 1, 2, \cdots, n)$ simultaneously. It may even be practicable to play all the games $\{\mathbf{G}_{ij}\}$ $(i, j = 1, 2, \cdots, n)$ simultaneously, using not only the full route $i_0 \to i_1 \to i_2 \to \cdots \to i_{k-1} \to j$, but also each partial route $i_r \to i_{r+1} \to \cdots \to i_{k-1} \to j$ as a separate play. In any case the arithmetic can be reduced by accumulating the totals of V_ρ for a series of plays and dividing by Np_j at the end, to obtain the average payment after N plays.

If $a_{ij} \geqq 0$ and $\sum_{j=1}^{n} a_{ij} < 1$, one can take $p_{ij} = a_{ij}$ and $v_{ij} \equiv 1$. Then $V_\rho \equiv 1$, and the computer has only to make a series of plays and keep a tally of the frequency with which the STOP ball was drawn in U_j $(j = 1, 2, \cdots, n)$.[4] If N is selected to be a power of 10, the only arithmetic required

to invert B is n divisions of a vector by p_j. When $v_{ij} = 1$, $A = R$ and $T = B^{-1}$, and hence $\sigma_{ij}^2 = p_j^{-1}(B^{-1})_{ij}[1 - p_j(B^{-1})_{ij}]$, corresponding to the variance of the binomial distribution.

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[1] The preparation of this paper was sponsored (in part) by the Office of Naval Research.
[2] A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, New York, 1946, p. 59. The writers are indebted to T. E. Harris for this reference.
[3] The fact that $\sigma_{ij} = \infty$ does not interfere with the convergence of the average value of N games to $(B^{-1})_{ij}$. However, conventional error estimates in terms of variances no longer apply and, in at least certain matrix inversions where $\sigma_{ij} = \infty$, the accumulated payment after N games cannot be normed so as to be asymptotically normally distributed as $N \to \infty$. See W. Feller, "Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung," *Mathematische Zeitschrift*, v. 40, 1935, p. 521–559 and v. 42, 1937, p. 301–312, and "Über das Gesetz der grossen Zahlen," Szeged, *Acta Univ., Acta Scient. Math.*, v. 8, 1937, p. 191–201.
[4] It is this case which we learned from von Neumann and Ulam.

Maximum-Interval Tables

Both the article by Herget & Clemence [*MTAC*, v. 1, p. 173–176] and the note by Miller [*MTAC*, v. 1, p. 334] on optimum-interval tables ignore the possibility of a continuously variable interval. It is of some interest to examine the reduction in the number of entries made possible by what might, by analogy, be termed "maximum-interval" tables. Using the principles of optimum-interval tables, with the modifications suggested by Miller, the tabular values of the argument are no longer restricted to terminating decimals so that the interval may be allowed to assume at each point the maximum value consistent with the stated allowable error.

The chief objection to a punched-card table in this form is that all (or nearly all) of the digits in the argument will have to be used in the interpolating factor. In some cases this objection could be overcome by inserting additional cards corresponding to values of the argument terminating in the appropriate number of zeros, or by splitting the whole range into a number of sub-ranges, in each of which the allowable error is varied slightly to make tabular arguments coincide with the end-points; or, of course, by an additional operation of subtraction. Generally, however, the saving in cards is not worth the additional cost of preparation and the resulting complication.

Herget in *The Computation of Orbits* [see *MTAC*, v. 3, p. 418–9] gives an optimum-interval table of $x^{-3/2}$ using quadratic interpolation, with a note "This is the first time such a table has ever been printed for use with a hand calculating machine." A human computer can easily exercise the requisite judgment to use the continuously variable intervals of a maximum-interval table; in the simplest case it will involve nothing more serious than allowing the interpolating factor occasionally to exceed unity in a particular digit. A punched-card machine can only do this at the expense of a separate operation. There may, therefore, be a use for variable interval tables in computation by desk machines.

The chief field of use is likely to be in connection with automatic digital computers, since the number of operations and the number of digits in the multipliers is often less critical than the amount of high-speed storage required. With cubic and quartic interpolation the number of entries required is very small indeed.

Before considering the theory of these tables, it is desirable carefully to examine the significance of "error" as applied to values obtained by interpolation. Corresponding to the function tabulated and the conditions of its use the error may be assessed in four ways: as an absolute or relative error in the tabulated function, or as relative to an absolute or relative error in the argument. Examples of these four cases are:

(a) $\sin x$, to a stated number of decimals; here the error is absolute.

(b) x^m, with a stated percentage error as is always required when multiplying factors are used to extend tables for a limited range of the argument; here the error is relative to the tabulated function.

(c) $\sin x$, with an accuracy corresponding to a stated absolute error (e.g., $0^\circ.0001$ or $1''$) in the argument.

(d) x^m, with an accuracy corresponding to a stated relative error (say 1 in 10^7) in the argument.

Each of these cases may occur and it is essential that the appropriate one should be adopted if the utmost economy in number of entries is to be achieved. Errors arise from several sources, but all except those due to approximations made in the interpolating formula can be substantially and effectively reduced by the retention of an additional, or guarding, figure; it will be assumed that this is always done in the type of table to be discussed, so that the theoretical errors of approximation may be directly equated with the allowable, or stated, errors.

The TAYLOR series for a function y of x at $x = a + t$ may be replaced, in the range $-\frac{1}{2}h \leqslant t \leqslant +\frac{1}{2}h$, by a series in terms of CHEBYSHEV polynomials:[1]

$$(1)\quad y(a+t) = \alpha_0 + \alpha_1 C_1(4t/h) + \alpha_2 C_2(4t/h) + \cdots + \alpha_p C_p(4t/h) + \cdots$$

where $C_p(\xi) = 2\cos(p \arccos \frac{1}{2}\xi)$ is the Chebyshev polynomial of degree p and where

$$(2)\quad \alpha_p = \frac{(\frac{1}{4}h)^p}{p!}\left\{y^{(p)} + \frac{(\frac{1}{4}h)^2}{1!(p+1)}y^{(p+1)} + \frac{(\frac{1}{4}h)^4}{2!(p+1)(p+2)}y^{(p+2)} + \cdots\right\}$$

in which $y^{(p)}$ is the p^{th} derivative of y at $x = a$.

It is known that the truncation of (1) after the n^{th} term will provide the most efficient polynomial approximation of degree $(n-1)$ to the function $y(a+t)$ in the range $-\frac{1}{2}h \leqslant t \leqslant +\frac{1}{2}h$. The leading term of the maximum error of the approximation is clearly $2\alpha_n$, which may for the present purpose be simplified to:

$$(3)\quad \frac{h^n}{n!2^{3n-1}}y^{(n)}.$$

This may now be equated to the allowable error to give the maximum permissible value of h as a function of x. If ϵ_1, ϵ_2, ϵ_3 and ϵ_4 are respectively the maximum absolute and relative errors allowable in y and x, then (3) must be equated with:

(a) ϵ_1 (b) $\epsilon_2 y$ (c) $\epsilon_3 y'$ (d) $\epsilon_4 x y'$

where the signs of the ϵ's are chosen so that h^n is positive.

Let z be an auxiliary variable which takes integer values corresponding to the values of x at which y is to be tabulated. Then, to a very good approximation, h may be identified with dx/dz so that we obtain the following expression for z in terms of x:

$$z = \int^x h^{-1}dx. \tag{4}$$

This will give z as a function of x, and in due course x, y and its derivatives as functions of z from which the table can be prepared.

For a simple power of x,

$$y = x^m \text{ (}m\text{ not a positive integer);}$$

substitution in and integration of (4) with the condition that $z = 0$ at $x = 1$, gives:

$$(5)\quad \begin{aligned} &\text{(a)}\quad z = \epsilon_1^{-1/n}(n/m)b(m, n)(1 - x^{m/n}) \\ &\text{(b)}\quad z = \epsilon_2^{-1/n}b(m, n) \ln x \\ &\text{(c)}\quad z = (m\epsilon_3)^{-1/n}nb(m, n)(1 - x^{1/n}) \\ &\text{(d)}\quad z = (m\epsilon_4)^{-1/n}b(m, n) \ln x \end{aligned}$$

where

$$\pm\, b(m, n) = [m(m - 1) \cdots (m - n + 1)2^{1-2n}/n!]^{1/n},$$

the sign being taken to make z positive. Cases (b) and (d) are, of course, identical in form. These expressions enable the number of entries for a particular range of x to be determined at once.

For example, in the seven-decimal table of reciprocals considered by Herget & Clemence the value of ϵ_1 is taken as $5 \cdot 10^{-7}$. A more realistic arrangement would arise from specifying maximum relative errors; a comparable table should therefore be based on $\epsilon_2 = \epsilon_4 = 5 \cdot 10^{-13/2}$, the geometric mean of the relative errors at beginning and end. Case (c), of a stated maximum absolute error in x, may well arise in practical computation; ϵ_3 can be taken as $5 \cdot 10^{-7}$. The corresponding expressions for z are:

$$\begin{aligned} \text{(a)}\quad & z = \tfrac{1}{4}n(4 \cdot 10^6)^{1/n}(1 - x^{-1/n}) \\ \text{(b), (d)}\quad & z = \tfrac{1}{4}(4 \cdot 10^{11/2})^{1/n} \ln x \\ \text{(c)}\quad & z = \tfrac{1}{4}n(4 \cdot 10^6)^{1/n}(x^{1/n} - 1) \end{aligned}$$

The number of entries required in a table to cover the range $x = 1$ to 10, with $n = 2$, 3, 4 (corresponding to linear, quadratic and cubic interpolation), are:

	$n = 2$	$n = 3$	$n = 4$
(a)	685	65	21
(b), (d)	648	64	21
(c)	2163	139	36

The reduction to 685, for case (a), as compared with the 1368 of Herget & Clemence and the 924 of Miller is substantial. There is here little difference between the requirements of cases (a), (b), and (d), though (b) is certainly the more realistic table. The very considerable reduction effected by the inclusion of a quadratic term would suggest that it might well be worth while introducing the additional step into the interpolation.

As a second example, consider a table of $x^{-3/2}$ with a maximum relative error of 10^{-7} which corresponds approximately to that of Herget's table. The number of entries (case (b)) required are:

$n = 2$ (linear) 3526.
$n = 3$ (quadratic) 204, as compared with Herget's 316.
$n = 4$ (cubic) 50.

The coefficients of the interpolating polynomial of degree $(n - 1)$ are obtained by expanding the individual terms of the truncated series (1) and rearranging as a power series in t. This series is then appropriate to the range $-\frac{1}{2}h \leqslant t \leqslant \frac{1}{2}h$. In order to facilitate computation it is desirable still further to transform the series as a power series in s where

$$s = (a + t) - x_0$$

and x_0 is a value of x terminating with several zero digits. It is only necessary to retain sufficient figures in the coefficients of powers of s to cover the range appropriate to t, provided they are rigorously consistent with the coefficients in the t-series; they should be calculated one at a time starting with that of s^{n-1}, which is of course the same as that of t^{n-1}.

It will be noted that for $y = x^m$, y in case (a) and x in case (c) are both polynomials in z of degree $n - 1$. In cases (b) and (d) both x and y are of exponential form; since the coefficient of $\ln x$ in (5) does not change rapidly for small changes in m, it would be possible to use the same series of values of x (or of y) for a number of fractional powers.

In his note Miller referred to the previously accepted convention of not modifying function values to reduce the error due to neglect of second differences. Actually such modification (for a slightly different reason) is in use in both the British and American surface and air almanacs; but in many cases there are sound reasons why the tabular values should be more accurate than the interpolates. It is possible to reduce considerably the error due to neglect of second differences without modifying the function values: in principle, the first difference is modified so that the error at the end of the range is numerically equal, with opposite sign, to the maximum error in the range. Specifically $\frac{1}{4}(3 - 2\sqrt{2}) = 0.043$ of the double second difference is subtracted from the first difference; the error is then -0.043 of the double second difference at the end of the interval and $+0.043$ at a point 0.414 along the interval. This error compares with 0.0625 without modification and 0.03125 with modification of the function. The device has obvious disadvantages, but it has some valuable applications—mainly in navigational tables.

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[1] This development is taken, with only slight change of notation, from Miller's paper "Two numerical applications of Chebyshev polynomials," R. Soc. Edinburgh *Proc.* Sec. A, v. 62, 1946, p. 204–210. It is, of course, simple to obtain the optimum expansions for small values of n by direct methods.

Planning and Error Considerations for the Numerical Solution of a System of Differential Equations on a Sequence Calculator

1. Introduction.—This paper deals with the solution of a specific system of fourteen ordinary differential equations,

$$z_i' = f_i(z_1, \cdots, z_{14}, t), \qquad \text{where} \qquad i = 1, 2, \cdots, 14. \tag{1}$$

The system was solved for the United States Navy Special Devices Center on the IBM Selective Sequence Electronic Calculator. A discussion of basic theoretical planning and error evaluation will be presented herein. The complete solution process will be described in a joint report with PAUL BROCK of the Reeves Instrument Company, which it is hoped will be available from the Special Devices Center. The present theoretical discussion outlines reasons for the choice of method, the detailed step-by-step process decided upon, the considerations which led to choice of interval length, and those which are concerned with the choice of scale.

There were three types of error in the solution process, namely, the truncation error, the rounding error, and the error due to the presence of nonanalytic functions in the f's. It is possible to obtain the immediate error of each type and to develop a uniform method of evaluating the complete effect of each such error. In the case of the truncation error, there is a resonance effect, a build-up of error, which is different in nature from instability, as this system is stable in the usual sense. The rounding error is treated by probability methods. However, the present treatment of this error is somewhat simplified since the effect is small. The nonanalyticities require a special technique, which will be discussed later. The methods developed in this paper, with this specific example in mind, differ considerably from those of RADEMACHER.[1]

2. Equations.—The f_i's in this system can be calculated from a given set of values, $z_1, \cdots, z_{14}$, and t by means of the elementary operations of arithmetic and the functions: square root, absolute value, sine, cosine, and arcsine. A series expansion was used for the arcsine; Newton's Method was used for the square root; and a table was used for the sine and cosine evaluation. The nonanalyticities appeared because of expressions which occurred in the form, $|z_i - k|$, when certain of the dependent variables, z, pass through specific values. The initial values were in a region of analyticity, but nonanalyticities occurred eleven times during the solution process.

3. General Method.—Consider f_j as a function of time. Suppose f_j has been evaluated at $t - h$, $t - 2h$, $t - 3h$, and $t - 4h$. A cubic in time is passed through these values and the polynomial integrated between $t - h$ and t. The result is a value for Δz_j, the increment in z_j between $t - h$ and t. In the usual Milne method, the value, z_j, obtained at t is substituted in the f's and a new cubic formed which goes through the values of f at t, $t - h$, $t - 2h$, and $t - 3h$. This in turn is integrated between $t - h$ and t to yield a new estimate for Δz_j. The new value, z_j, can be used in the same way to obtain another value of Δz_j and the process continued until there is no

significant difference between successive values of z_j. A modification of this method is used. The first step in the above process is referred to as the "open" integration; the remaining steps are called "closed" integration.

The value of Δz_j obtained from the open integration is a simple linear combination of the values of f_j for $t - h$, $t - 2h$, $t - 3h$, and $t - 4h$, i.e.,

$$(2)\quad \Delta z_j = \frac{h}{24}[55f_j(t-h) - 59f_j(t-2h) + 37f_j(t-3h) - 9f_j(t-4h)].$$

For a closed step

$$(3)\quad \Delta z_j = \frac{h}{24}[9f_j(t) + 19f_j(t-h) - 5f_j(t-2h) + f_j(t-3h)].$$

4. Step-by-Step Error.—For simplicity, let $t = 0$ in the above equation. The usual remainder argument shows that, if P is a polynomial of the third degree which equals f_i at $t = -h, -2h, -3h$, and $-4h$, then for $-h \leqq t \leqq 0$

$$(4)\quad f_i - P = f_i^{(\mathrm{IV})}(x')\frac{(t+h)(t+2h)(t+3h)(t+4h)}{4!}$$

for some x' between $-4h$ and 0. Suppose $f_i^{(\mathrm{IV})}$ is a constant and integrate between $-h$ and 0; then, the error, $\epsilon_i^{(0)}$, in Δz_i of the open integration is

$$(5)\quad \epsilon_i^0 = .349f_i^{(\mathrm{IV})}h^5.$$

If the correct value for f_i at $t = 0$ were known, a similar argument for the closed integration step would yield the error in the closed steps, i.e.,

$$(6)\quad \epsilon_i^c = -.0264f_i^{(\mathrm{IV})}h^5.$$

However, in the first closed integration step for Δz_i, the following expression holds:

$$(7)\quad \Delta z_i^{c1} = \frac{h}{24}[9f_i^{(0)}(t) + 19f_i(t-h) - 5f_i(t-2h) + f_i(t-3h)],$$

where $f_i^{(0)}(t)$ is obtained by substituting the values of z involving the open integration error. Consequently, the first closed integration step involves an error

$$(8)\quad \epsilon_i^{c1} = \epsilon_i^c + \frac{9h}{24}\sum_{\alpha=1}^{14}\frac{\partial f_i}{\partial z_\alpha}\epsilon_\alpha^0.$$

Now, if h is sufficiently small, the second term is less important than the first. Consequently, if $f_i^{(\mathrm{IV})}$ is essentially constant, ϵ_i^0 and ϵ_i^{c1} differ in sign, and the values for the open integration and the first closed integration for Δz_i span the true value. The difference between these values forms an upper bound and a good estimate for the truncation error.

It may be noted that the same argument shows that the error, ϵ_i^{ck}, in the k^{th} step can be expressed in terms of the error, ϵ_i^{ck-1}, in the previous step, i.e.,

$$(9)\quad \epsilon_i^{ck} = \epsilon_i^c + \frac{9h}{24}\sum\frac{\partial f_i}{\partial z_\alpha}\epsilon_\alpha^{ck-1}.$$

This forms an easy basis for convergence considerations. The final convergence has an error which satisfies the equation,

$$\epsilon_i^{c\infty} - \frac{9h}{24}\sum \frac{\partial f_i}{\partial z_\alpha}\epsilon_\alpha^{c\infty} = \epsilon_i^{e}. \tag{10}$$

5. Actual Method Used.—The f's are complicated functions from the computational point of view, and their evaluation consumes much machine time. With this in mind, consider the relatively simple method given above. In view of equation (10) and the machine time required to evaluate the f's, there seems to be no point in carrying through the iterative process to the bitter end. In fact equations (8) and (10) suggest that considerations can be confined to an open and closed step. However, the price of an open and closed step in machine time is the same as the price of two open steps. It is possible to compare the error which follows from an open and closed step with interval, h, and the error of two open steps with interval, $\frac{h}{2}$. The latter can be obtained by dividing .349 in equation (5) by 16 and comparing the result, .022, with .0264 of equation (6). Thus, the open integration procedure, which is simplest from the coding point of view, is the best. (This argument is based on the use of cubic approximations.) Even if this situation were reversed, it would be better to choose the open integration method since a smaller step interval is used, and, hence, error estimates are more dependable. Similar objections may be raised to any other system which uses a larger h.

However, certain other considerations must be made. The choice of the interval, h, is necessarily based on the initial information given in the problem. Later in the run, h may be too large or too small. This can only be established by knowing the truncation error, which can be indicated by the difference between a closed and open step, as has been shown above. The procedure is to use an open integration at each step but, at every tenth step, to follow it by a closed integration. At each tenth step, the difference between the open and closed steps can be obtained from the printer in the sequence calculator, and one can infer from this whether the truncation error is tolerable.

If the truncation error is small enough, one is justified in increasing h by an integral multiple. This is readily done using the previously computed values of f. The coding for the problem was set up in such a way that one could start by feeding in cards with the properly spaced values of f for four points. To increase h then by an integral multiple, it was only necessary to take certain cards previously punched by the machine and feed them in. It was not necessary to decrease h, but if necessary, intermediate values of f could have been obtained by interpolation involving four differences.

6. Choice of h.—From physical reasoning and also from an actual consideration of the variational equation at the initial part of the run, it was determined that the system of equations is stable in the absolute sense. When an error is made, the effect of this error does not increase indefinitely, but, instead, dies down at a very slow rate.

Initially, the possibility of a run to $t = 100$ had to be planned for. Calculations based on the initial situation and these considerations indicated that

the interval, $h = .04$, is too large and that $h = .02$ would be about right. But the cost of the solution for $h = .02$ was prohibitive. Furthermore, it was felt that the assumption that all the errors would build up in the same direction was too conservative, and, hence, it was decided to use the interval, $h = .04$. Due, however, to the resonance effect which will be discussed later, the errors do effectively build up in the same direction so the stated reason is not sound. On the other hand, $f^{(IV)}$ does decrease as evidenced by the truncation error, and, while the total accuracy was not quite that planned, the result was adequate.

To begin the process, it is necessary to compute the values of f's at four points. These initial values were computed by finding the corresponding values of z by means of Taylor's series and computing the f's. Since it is relatively easy to increase h, an initial value of $h = .001$ was chosen, and the f's were computed for $t = .000$, $t = .001$, $t = .002$, and $t = .003$. Using $h = .001$, the computation process was carried to $t = .100$. This resulted in values at $t = .00$, $t = .02$, $t = .04$, and $t = .06$, which were used to start the run with $h = .02$. At $t = .08$ the values for z using the short interval, $h = .001$, and also an open step value for $h = .02$ and a closed step value were then available. Of course, the smaller h value yields a much more accurate value for z. It is interesting to observe that for every z, the accurate result lay between the open and closed values.

The $h = .02$ run was continued until about $t = 2.18$, and the $h = .04$ run began with $t = 2.10$. Again an overlap was permitted for checking purposes in order to be sure that the change of interval was correctly made. It was found that the shorter interval value again lay between the open and closed value for the longer interval.

The interval, $h = .04$, was continued until $t = 68$, where one of the variables, z, ran off scale, and the solution was no longer of interest. During an earlier false run, an attempt was made to increase the interval to $h = .12$, but this yielded large truncation errors. Therefore, the interval, $h = .04$, was used again. During the final true run, no such increase was attempted.

After the $h = .04$ run, the $h = .02$ run was repeated and continued in order to check the error of the $h = .04$ run. Unfortunately, only the run from $t = 2.10$ to $t = 2.90$ was duplicated at an interval, $h = .02$. However, this provided a valuable method of checking the error theory developed later.

7. Scaling Questions and Watches.—The registers in which a number can be entered contain 9 digit-positions. The scale for a quantity must be chosen so that the effect of truncation error can be clearly seen. On the other hand, some leeway against spill-over must be available. Consequently, the size of each quantity must be estimated in advance.

For this estimation, a certain amount of physical information is available. A set of assumptions, based on this information, were made concerning the sizes of the quantities involved, and values for all other quantities were estimated. However, the assumptions were not based on completely reliable information, and it was necessary to verify that, during the course of the computation, these assumptions remained valid. This was done by a system of "watches." Certain of the quantities were printed by the machine, and others could be obtained readily from printed results. During the complete calculation, up to the time the calculation was stopped, the assumptions held.

8. Hand Computations.—The Taylor's series for the z's had to be obtained by hand computation. This was done in triplicate, as it was believed that the error rate was approximately about 1 in a 100 calculation steps. Since the amount of work was of the order of magnitude of 10,000 steps, a calculation in duplicate would have too high a probability for a duplicate error.

In addition, it was also necessary to provide a hand computation of two steps of the machine computation for checking purposes. This was done for the $t = .004$ and $t = .005$ steps, but unfortunately many values appeared as zeros necessitating the provision of further material for checking purposes. In addition, a complete machine step consisting of an open and closed step at the later value, $t = 6.44$, was duplicated by hand and the values checked against the machine values. In these checking computations, the

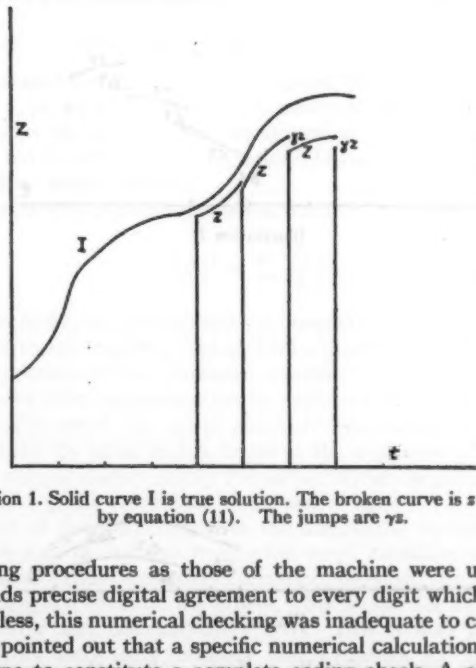

Illustration 1. Solid curve I is true solution. The broken curve is z as defined by equation (11). The jumps are γz.

same rounding procedures as those of the machine were used. Hence, a check demands precise digital agreement to every digit which appears.

Nevertheless, this numerical checking was inadequate to catch one error. It should be pointed out that a specific numerical calculation is logically of too low a type to constitute a complete coding check. A complete check should start with the coding and reconstruct the original mathematical relations. Thus, it is clear that a correct coding system is required.

9. Definition of the Error.—It has been mentioned that there are three types of errors inherent in the solution. Associated with every error there are two problems. It is necessary to estimate the error at the point where it is made. It is also necessary to evaluate the effect of the error on those portions of the solution which follow in time the point at which an error is made. Let the vector $z = (z_1, \cdots, z_{14})$ denote the computed solution. Let $(cz_1, \cdots, cz_{14})$

denote the correction vector which, when added to z, will yield the true solution. The computed solution exists only at the points at which a computation was made. Let z be defined between such points by the condition that it satisfies the equations

$$z_u' = f_u(z_1, \cdots, z_{14}, t) \tag{11}$$

and is continuous at the left-hand end point of each such interval. At the right-hand end point of any interval, z has a discontinuity corresponding to the error made at the right-hand point. (See Illustration 1.)

At each solution point the total error is made up of two parts. The first part, termed the rounding error, is the difference between the computed

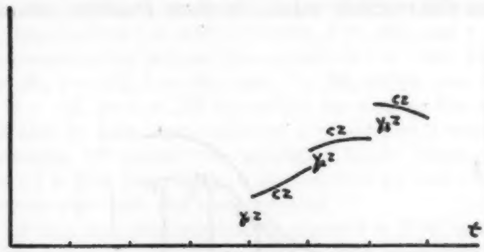

Illustration 2.

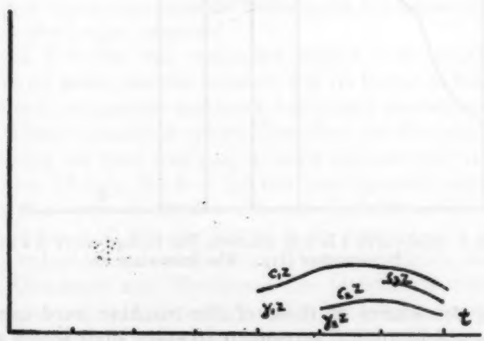

Illustration 3.

solution and the computationally perfect result of the same computation made with no rounding errors. The second part is the difference between the computationally perfect result and the result of a correct integration between the previous step and the present one. If the points involved in the integration process do not span a point of nonanalyticity for the f being integrated, the second part will be termed the truncation error. If, on the other hand, there is a point of nonanalyticity, for the f being integrated, present in the span covered by the integration process, this error will be called a nonanalyticity error. The distinction between truncation errors and non-

analyticity errors is set up purely with the situation of the present practical problem in mind and appears to be the most convenient in this case.

The total correction vector at the α computational point corresponding to the total error made at that point is denoted by $\gamma_\alpha z$; and the truncation correction, the rounding correction, and the nonanalyticity correction vectors for the corresponding errors are denoted by $\gamma_\alpha' z$, $\gamma_\alpha'' z$, and $\gamma_\alpha''' z$, respectively.

10. Differential Equation for the Error.—Since $z + cz$ is the desired solution of the system of equations, the following relationship holds, i.e.,

$$(z_u + cz_u)' = f_u(z_1 + cz_1, \cdots, z_{14} + cz_{14}, t). \tag{12}$$

For the intervals between points of computation, z also satisfies the equations (11), and, consequently,

$$(cz_u)' = f_u(z_1 + cz_1, \cdots, z_{14} + cz_{14}, t) - f_u(z_1, \cdots, z_{14}, t). \tag{13}$$

As z is now considered to be known, equation (13) is a differential equation on the vector, cz, which holds between points of computation. At points of computation, cz has a jump, γz, corresponding to the error made at this point. This jump condition and equation (13) determine precisely the correction, cz. (See Illustration 2.)

For practical computational reasons, equations (13) are made linear, i.e.,

$$(cz_u)' = \sum_v \frac{\partial f_u}{\partial z_v} cz_v. \tag{14}$$

The coefficients $\partial f_u/\partial z_v$ are regarded as constants over an interval which is large relative to the length of a step. Undoubtedly, this introduces an error into the error computation. However, consider the result of solving (14) as simply the zero order approximation to a solution of (13) by a PICARD iteration process. The use of (14) can be justified by estimating the corresponding solution of (13). By using higher terms in the expansion of the right-hand side of (13), one can readily justify the use of (14) for the purpose of attaining the real objective, the estimation of the size of cz_u.

One effect of the linearization of the differential equations is that the correction vector, cz, for the truncation error, rounding error, and nonanalyticity error can be separately computed, provided one error is assumed to have a negligible effect on the others. This is clear for an estimate of the truncation error and nonanalyticity error. This is also justified for the rounding errors since only maxima and variances for these errors are computed.

Thus, a truncation correction, $c'z$, can be defined as satisfying (14) between computation points and having a jump, $\gamma_\alpha' z$, at the α computation point. A similar definition holds for $c''z$ and $c'''z$, the rounding and nonanalyticity corrections. The sum $c'z + c''z + c'''z$ satisfies (14) between computation points and has the required jump, $\gamma_\alpha z$, at the α computation point.

A function, $c_\alpha' z$, is introduced which is zero before the α computation point, has a jump, $\gamma_\alpha' z$, at this point, and satisfies (14) after this point (see

Illustration 3). The definition of $c_\alpha''z$ and $c_\alpha'''z$ is similar, and one can readily show that

$$(15) \qquad c'z = \textstyle\sum_\alpha c_\alpha'z, \qquad c''z = \sum_\alpha c_\alpha''z, \qquad c'''z = \sum_\alpha c_\alpha'''z.$$

11. Error Effect Procedure.—In view of formula (15), the immediate problem is to compute $c_\alpha'z$. As was previously mentioned, intervals will be considered over which the equations (14) can be regarded as linear differential equations with constant coefficients. The following equation can then be written in vector form,

$$(16) \qquad (cz)' = Hcz.$$

This equation is solved by finding the characteristic roots, $\lambda_1, \cdots, \lambda_{14}$, of H and the corresponding characteristic vectors. Although zero is a double characteristic root, there are 14 characteristic vectors, $x^{(1)}, \cdots, x^{(14)}$. The general solution is in the form,

$$(17) \qquad y = \sum k_i e^{\lambda_i t} x^{(i)}.$$

Now $c_\alpha'z$ is the solution which at $t = t_\alpha$ has the value $\gamma_\alpha'z$. Let $X^{(1)}, \cdots, X^{(14)}$ be the set of dual vectors to the characteristic vectors $x^{(1)}, \cdots, x^{(14)}$, i.e.,

$$(18) \qquad x^{(i)} \cdot X^{(j)} = \delta_{ij}.$$

Then

$$(19) \qquad c_\alpha'z = \sum_i \gamma_\alpha'z \cdot X^{(i)} e^{\lambda_i(t-t_\alpha)} x^{(i)}.$$

Of course, the λ_i's, $x^{(i)}$'s, $X^{(i)}$'s assume complex values, but $c_\alpha'z$ will be real since $\gamma_\alpha'z$ is real. A similar formula holds for $c_\alpha''z$ and $c_\alpha'''z$.

In computing the λ_i's, a further assumption was made which permitted one to factor the determinant $|H - \lambda|$ into two quintics, a quadratic, and λ^2. The quintics were solved by finding a real root by NEWTON's Method, and the remaining quartics were solved by a method of LIN.[3]

The process by which the characteristic vectors and their duals were obtained was somewhat complicated, since it was desirable to avoid fourteenth-order matrix operations. However, this was specific to the problem and will not be discussed here.

12. Truncation Error.—In view of the above mechanism for evaluating the effect, $c_\alpha'z$, of an error, $\gamma_\alpha'z$, one has the problem of obtaining $\gamma_\alpha'z$ in such a form that the summation process of (15) can be effectively carried out. For $\gamma_\alpha'z$ the difference is found between the open and closed integration values at every tenth step. This was expressed in analytic form. However, there is a more effective way of obtaining $\gamma_\alpha'z$.

The f_i for each i is given. Furthermore, by inspection it is seen that each f_i can be expressed approximately in the form

$$(20) \qquad f_i = f_i^0(t) + a_1 e^{-\alpha_1 t}\cos(\beta_1 t + \gamma_1) + a_2 e^{-\alpha_2 t}\cos(\beta_2 t + \gamma_2) + a_3 e^{-\lambda_0 t},$$

where $f_i^0(t)$ is a slowly changing function, while $\alpha_1 + i\beta_1$, $\alpha_2 + i\beta_2$, and λ_0 are roots of the characteristic equations for H. The truncation correction

which is needed when an open integration is performed on the function, f, is

$$(21) \quad \int_t^{t+h} f dt - \frac{1}{24} h[55f(t) - 59f(t-h) + 37f(t-2h) - 9f(t-3h)].$$

This expression for the correction is linear, and if f is expressed as a linear combination of terms in the form $Ae^{-\lambda t}$ (where λ is complex), then for each such term the error is

$$(22) \quad hAe^{-\lambda t}\left[\frac{i - e^{-\lambda h}}{\lambda h} - \frac{1}{24}(55 - 59e^{\lambda h} + 37e^{2\lambda h} - 9e^{3\lambda h})\right].$$
$$= Ae^{-\lambda t}h\Gamma_0(\lambda h).$$

Thus, in general, if

$$f = \sum_{i=1}^{N} a_i e^{-\lambda_i t},$$

the truncation correction is

$$(23) \quad \gamma z = \sum_{i=1}^{N} h a_i e^{-\lambda_i t}\Gamma_0(\lambda_i h).$$

From the point of view of frequency analysis, $\Gamma_0(\lambda h)$ is exceedingly interesting. For, if $\Gamma_0(u)$ is expressed as a power series in u, we obtain

$$(24) \quad \Gamma_0(u) = .349u^4 + .716u^5 + .789u^6 + .613u^7 + .375u^8 + .191u^9 + .084u^{10} + \cdots \text{ (see footnote 3).}$$

This reflects the fact that the method of integration used was set up to integrate cubic polynomials perfectly. From the frequency standpoint, this means that there is a fourth-order contact at $\lambda = 0$. On the other hand, this suggests an investigation of other integration formulae which would be precise at certain other frequencies particularly chosen for the problem at hand. Such an investigation has been initiated with some interesting results. For instance, an arbitrarily good approximation cannot be found over a connected λh interval of length exceeding 2π no matter how many constants are available in the integration formula; but for a smaller interval such an approximation can be found.

The above formula indicates that the contribution of the low frequency terms to $\gamma' z_i$ is negligible, and indeed only one term makes an effective contribution to $\gamma' z_i$. Consequently, one can obtain $c_\alpha' z$ by (23) and (19). Furthermore, by a summation, based on a geometrical progression, it is possible to find $c' z_j$ using (15).

Since the same frequency appears in both (23) and (19), it is noted that when the summation (15) is carried out, a term in the expansion of cz_α is obtained in the form,

$$(k_1 t + k_2)e^{-\alpha_1 t}\cos(\beta_1 t + \gamma),$$

while the other terms are exponentials without t in the coefficients. As far as this term is concerned, the total error builds up until the rather small exponential, α_1, eventually cuts it down. This was the important term in the error analysis.

Thus even though the errors, $\gamma_\alpha' z$, oscillate in sign, their total effect builds up. The result is analogous to the solution of a differential equation,

$$L(y) = g(x),$$

where $L(y)$ is a linear differential operator with constant coefficients, whose indicial equation has roots with negative real parts and in which $g(x)$ contains a term corresponding to one of these roots. This phenomenon produces a resonance effect.

This resonance phenomenon will occur whenever the original system of equations (1) and the variational equation (16) are close enough so that terms with frequencies associated with the variational equations appear in the f's of (1) regarded as functions of the time. If practical, the method of integration should discriminate against these frequencies.

As was mentioned previously, a stretch from $t = 2.10$ to $t = 2.90$ was available over which there is a solution corresponding to an interval of $h = .04$ and one corresponding to $h = .02$. It is readily seen from the above that the error for the smaller interval should be approximately $\frac{1}{16}$ that for the larger interval. Now the theoretically computed estimate of the error can be compared with the difference between these solutions. The agreement between the two curves was very good in both size and shape.

Since the error is a solution of a system of linear equations, the improvement in $\gamma_\alpha' z$ obtained by halving the interval will be reflected in $c_\alpha z$. Thus, halving the interval is an excellent method of estimating the error when h is small enough to justify the linearization upon which (14) is based. Unfortunately, the cost in machine time is tripled by this procedure.

13. Rounding Error.—To find the correction vector, $\gamma'' z$, for the rounding error, it is necessary to compare the correct calculation with the actual calculation. Naturally, this involves a study of the computation process used for each f. The correction, $\gamma_\alpha'' z_j$, for the j^{th} component is made up of a sum of a number of terms, a few of which are periodic, but most of which can be regarded as chance variables. Since the total effect of rounding error in this problem is not great compared with the other errors, it was felt that certain simplifications were justified. Hence, the $\gamma_\alpha'' z$ was considered to be in the form

$$\gamma_\alpha'' z = (a_1\epsilon_1^\alpha, a_2\epsilon_2^\alpha, \cdots, a_{14}\epsilon_{14}^\alpha), \tag{25}$$

where the a_j is a maximum value for the component involved and the ϵ_j^α's are independent evenly-distributed chance variables taking on values between -1 and 1. However, in the analysis, $a_j\epsilon_j^\alpha$ could be replaced by a more precise expression for this component if it were necessary.

It follows from (25), (9), and (15), that the total effect of the rounding error is in the form

$$c'' z = \textstyle\sum_{\alpha, j} E_{\alpha, j}(t)\epsilon_j^\alpha. \tag{26}$$

The maximum can be estimated by proper summation methods; and it is also possible using SCHWARZ's inequality and these same summation methods to obtain a bound for the variance of $c'' z$.

14. Points of Nonanalyticity.—In those cases in which an f_i was nonanalytic, it could be written in the form

$$f_i = F_i + H_i \operatorname{sign}(z_j - A), \tag{27}$$

where F_i and H_i are analytic and A is a constant. In general, $i \neq j$, and it was a relatively simple matter to compute both the correct integral for the term, $H_i \operatorname{sign}(z_j - A)$ and the value obtained from the method of integration used. There is a discrepancy due to the fact that this method of integration assumes that the integrand is a polynomial while the term $H_i \operatorname{sign}(z_j - A)$ is in general discontinuous. However, this difficulty persists over four points, and the successive errors obtained in this way tend to cancel. Therefore, except for a temporary disturbance, the effect of an individual nonanalyticity is slight. Since there are relatively few of these, the total effect of the nonanalyticities is readily calculated by means of (19) and (15).

15. General Planning Considerations.—Experience suggests that the preliminary analysis of a problem of this sort be based on the matrix, $H = ((\partial f_i/\partial z_\alpha))$, which, of course, can be evaluated initially. The characteristic roots of H indicate the frequencies present. The total error permitted can be divided into three not necessarily equal parts, one for truncation, one for rounding, and one for miscellaneous. In view of the resonance error, the total truncation error should be divided by the expected length of the computation to obtain the error permitted per unit interval, ϵ_0, provided the situation is stable. If the open integration formula is used and if one λ, λ', predominates among the characteristic roots of H, then h, the interval width to be used initially, can be obtained from

$$\epsilon_0 = \Gamma_0(\lambda' h). \tag{28}$$

If more than one λ is important in H from this point of view, ϵ_0 will have to be further subdivided and a portion allocated to each effective root. If some λ has positive real parts, i.e., if the situation is unstable, it is usually permissible to make a change in scale and a similar discussion applies.

After having left the region around the initial value, h can be determined by comparing the result of an open and closed integration at specified intervals. In this connection, it should be pointed out that the quantity, $\Gamma_c(\lambda h)$, which determines the error for a closed integration in the same way as Γ_0 did for an open integration, is given by

$$-(.0264u^4 + .0341u^5 + .0246u^6 + .0129u^7 + .0055u^8 + .0020u^9 + .0006u^{10} + \cdots). \tag{29}$$

The change in sign again indicates that the results of an open and closed integration span the true value, and it is apparent that the difference is an excellent estimate of the truncation error.

16. Conclusion.—It seems clear from the above that a plan of solution should be based on considerations of over-all error rather than upon consideration of error per step. One such plan is given. This plan indicates a method for choosing h initially and also during the course of the solution. The ideas used can be supplemented by algebraic methods to estimate the total error when the solution process has been completed.

Our experience involved a stable system of differential equations, but the use of frequency analysis is justified in most cases, including unstable cases in which a scale change occurs.

F. J. M.

[1] Harvard University, Computation Laboratory, *Annals*, v. 16, p. 176–187. [*MTAC*, v. 3, p. 437.]
[2] Shih-Nge Lin, "Numerical solution of complex roots of quartic equations," *Jn. Math. Phys.*, v. 26, 1947, p. 279–283.
[3] This formula assumes that the computed and hence available f is the function to be integrated. If one wishes to make an allowance for the distinction between this f and the correct f associated with the true solution, a difference equation must be solved. In case λ is a characteristic root of H, the corrected value Γ_0 is obtained by dividing the above formula (24) by

$$1 + .5000u + .1667u^2 + .0417u^3 - .0972u^4 - .1424u^5 \cdots.$$

This correction is significant.

RECENT MATHEMATICAL TABLES

761[A].—R. Coustal, "Calcul de $\sqrt{2}$, et réflexion sur une espérance mathématique," Acad. Sci. Paris, *Comptes Rendus*, v. 230, 1950, p. 431–432.

The first four terms of the binomial expansion of

$$\sqrt{2} = a(1 - 2x)^{-\frac{1}{2}}$$

where

$$a^2 = 2 - 4x$$

and a is an approximation to $\sqrt{2}$, good to 333D, were used to obtain $\sqrt{2}$ to 1032D. Besides this value the author gives the distribution of digits in the 1033S values of $\sqrt{2}$ and $1/\sqrt{2}$. In the first 1000D in the $\sqrt{2}$, the digits 0–9 have the following frequencies

108, 98, 109, 82, 100, 104, 90, 104, 113, 92.

Such a distribution has a chi-square of 8.38. The probability of such a value from a normal distribution is almost exactly 1/2. For $1/\sqrt{2}$ the probability is merely .05. [Compare MTAC, v. 4, p. 109–111].

The author "reflects" on the paradox that if one takes the product of the first 1033 digits of the decimal expansion of a real number x in the interval $0 < x < 1$, the expected value of the product is $(9/2)^{1033} > 10^{674}$, whereas the probability that it is exactly zero is $1 - (9/10)^{1033} > 1 - 10^{-47}$.

D. H. L.

762[C].—H. S. Uhler, "A mathematician's tribute to the state of Israel," *Scripta Mathematica*, v. 14, 1949, p. 281–283.

The author gives ln 173 and ln 5709 to 290D.

763[D, H, L].—C. N. Davies, "The sedimentation and diffusion of small particles," R. Soc. London, *Proc.*, v. 200, 1949, p. 100–113.

This paper contains a table of the first 16 positive roots of the equation

$$2\alpha x + \tan x = 0$$

for $\alpha = 2, 1/2, 1/3, 1/6, 1/30$. Values are to 5S, 6S and 7S. For $\alpha = 1/2$ the equation is $\tan x = -x$ and the first 11 roots given in this case agree with those of POOLER, *MTAC*, v. 3, p. 496.

764[D].—TOSHIZO MATSUMOTO, "On Hayami's turbulent tensor," Kyoto, Imperial University, College of Science, *Memoirs*, v. 24A, no. 2, 1944, p. 63–72.

On p. 66–69, is a table, computed by HIROSI NAKAHETA, of the functions: $z^{\frac{1}{2}} \cos \frac{1}{2}\pi z$, to 4D, δ; $z^{\frac{1}{2}}$ to 8D; $\cos \frac{1}{2}\pi z$ to 7D, for $z = 0(.01)2$.

R. C. A.

765[E].—H. G. HOPKINS, "Elastic deformations of infinite strips," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 164–181.

The appendix (p. 181) is a 6D table of

$$g_1 = (16 \cosh^2 x)/(55 \cosh^2 x + 25x^2 + 9)$$

and

$$g_3 = (16 \cosh^2 x)/(39 \cosh^2 x + 9x^2 + 25)$$

for $x = 0(.1)6$. At $x = 6$ the functions are already near their limiting values 16/55 and 16/39.

766[F].—D. JARDEN & A. KATZ, "Additional page (477) to D. N. Lehmer's Factor Table," *Riveon Lematematika*, v. 3, 1949, p. 49 [English summary p. 52].

This is the same as UMT **85**[F], *MTAC*, v. 4, p. 29.

767[F].—A. KATZ, "Some more new factors of Fibonacci-numbers," *Riveon Lematematika*, v. 3, 1949, p. 14 [English summary, p. 54].

The author continues the factorization of the terms of the Fibonacci sequences U_n and V_n [see *MTAC*, v. 3, p. 299] giving the complete factorization of $U_{117} = 2 \cdot 233 \cdot 29717 \cdot 135721 \cdot 673024656781$, $V_{73} = 151549 \cdot 11899937029$ and $V_{108} = 2 \cdot 7 \cdot 23 \cdot 6263 \cdot 103681 \cdot 177962167367$. The factors 128621 and 119809 are given for V_{109} and V_{128} respectively. No further factors of the two series exist below $2 \cdot 10^5$ up to $n = 128$.

D. H. L.

768[F].—G. PALAMÀ, "Tabella delle posizioni iniziali relative al 'Neocribrum' di L. Poletti," Parma, Univ., *Rivista Mat.*, v. 1, 1950, p. 85–98.

The "Neocribrum" is a form of factor table devised by L. POLETTI [*MTAC*, v. 3, p. 532] and has for column headings the set of 6 numbers

(S) 1, 7, 11, 13, 17, 19, 23, 29

which are those prime to 30. Two cells of the table in the same column and adjacent lines correspond to numbers differing by 30. Once a prime $p > 5$ appears as a factor of a number in any one column it reappears p lines farther down and continues to appear periodically. Hence to construct the factor table it is necessary to know the line number at which a given prime factor

will first appear. The purpose of the paper is to supply this information for all primes p for which $17 \leq p \leq 3547$ (the primes 7, 11 and 13 are already printed into the otherwise blank forms of the sieve). Hence the table gives for each of the 491 primes the solutions x_i of the congruences

$$30(x_i - 1) + m_i \equiv 0 \pmod{p} \qquad (i = 1(1)6)$$

where m_i are the numbers of (S). The awkward $x - 1$ is due to the fact that Poletti numbers his lines beginning with 1 rather than 0. The table gives also the least positive h for which $h + 1001$ is divisible by p. This is to enable the application of the results to successive "cycles" of the table.

It is a little hard to see the need for publishing such a table. If one is going to construct a factor table of just this kind it is clearly indispensable. However, the table has other uses. The column headed 29 gives directly the value of $1/30$ modulo p and the first column is a convenient list of primes.

D. H. L.

769[G].—F. N. David & M. G. Kendall, "Tables of symmetric functions—part I," *Biometrika*, v. 36, 1949, p. 431–449.

The authors present tables of coefficients in the linear representation of the monomial symmetric function in terms of products of sums of like powers, and conversely the coefficients in the expansion of the latter in terms of the former. The tables extend to symmetric functions of weights ≤ 12. Thus from the two tables of weight 4 we read, for example, that

$$2 \sum \alpha^2\beta\gamma = 2 \sum \alpha^4 - 2(\sum \alpha^3)(\sum \alpha) - (\sum \alpha^2)^2 + (\sum \alpha^2)(\sum \alpha)^2$$

and that

$$(\sum \alpha^2)(\sum \alpha)^2 = \sum \alpha^4 + 2 \sum \alpha^3\beta + 2 \sum \alpha^2\beta^2 + 2 \sum \alpha^2\beta\gamma.$$

In order to avoid fractions the monomial symmetric functions are multiplied by the product of factorials of the exponents in the partition of the weight represented by the monomial. The resulting functions are called "augmented." Thus $\sum \alpha^2\beta^2\gamma^2\delta\epsilon$ is multiplied by 3! 2!

The authors appear to be unaware of the tables of Sukhatme, Ziaud-Din, Kerawala and Kerawala & Hanafi [*MTAC*, v. 3, p. 24]. The first of these gives the same information for weights ≤ 8. The others give only half as much information (i.e., the coefficients in the expression of the monomials in terms of sums of like powers) for weights 9, 10 and 11. The table of weight 12 under review is completely new.

The various checking procedures used by the authors should be sufficient to produce a set of tables completely free from error. Nevertheless it would have been wise to collate the tables with those previously given which are known to contain errata [*MTAC*, v. 3, p. 24]. The application of the tables to problems in statistics is treated briefly.

The tables are arranged beautifully in lexicographical order, the two triangular halves fitting into a perfect rectangular layout. The type, though rather small, is quite clear.

D. H. L.

770[G].—J. A. TODD, "The characters of a collineation group in five dimensions," R. Soc. London, *Proc.*, v. 200, 1949, p. 320–336, insert between p. 336–337.

The paper gives a table of characters of a primitive group of collineations of order $6531840 = 2^8 \cdot 3^6 \cdot 5 \cdot 7$ in five space and a table of characters of a subgroup of index 2. The group is described in another paper.[1]

[1] J. A. TODD, "The invariants of a finite collineation group in five dimensions," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 73–90.

771[I].—ANDERS REIZ, "On quadrature formulae," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 119–126.

The author considers the usual quadrature formula for an integral with weight function $w(x)$:

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^n p_i f(x_i) + R_n.$$

As is well known, the best choice of x_i is the set given by the GAUSS' method corresponding to $w(x)$. However, these values are not rational and the computer is faced with the task of interpolating to find $f(x_i)$. If we choose for x_i the Gaussian values rounded off to 2D and modify the p_i accordingly we obtain a formula which is theoretically easier to use and which retains nearly the full force of the Gauss approximation.

Six tables of the x_i, p_i are given for the six weight functions

$$1, \quad \pi^{-1}(1-x^2)^{-\frac{1}{2}}, \quad \pi^{-1}(1-x^2)^{\frac{1}{2}}, \quad \pi^{-1}(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}},$$
$$e^{-x} \quad \text{and} \quad \pi^{-\frac{1}{2}} \exp(-x^2).$$

The order n extends as far as 10, 8, 8, 5, 5, and 8 in these respective cases.

For the first four weight functions the interval (a, b) of integration is either (0, 1) or $(-1, 1)$ while in the last two cases it is $(0, \infty)$ and $(-\infty, \infty)$ respectively.

The x_i are given to 2D, as mentioned above, and the coefficients p_i to 7D.

Five examples are given comparing the present method with the strict Gauss formula and WEDDLE'S rule. As might be expected, the modified Gauss method is only slightly inferior and considerably easier to use. It is much superior to Weddle's rule.

In "actual practice" the above avoidance of interpolation difficulties may be, in some cases, only apparent. In fact if the function to be integrated is observed or tabulated on an interval (a, b) different from (0, 1) or $(-1, 1)$, such as $(0, 2\pi)$, then a transformation of variable is required which may introduce interpolation after all.

D. H. L.

772[I].—HERBERT E. SALZER, "Formulas for complex cartesian interpolation of higher degree," *Jn. Math. Phys.*, v. 28, 1949, p. 200–203.

The purpose of the present table is to provide the coefficients for a Lagrangean interpolation polynomial adaptable to interpolation over a square grid in the complex plane. The polynomial assumes the form

$$f(P) = \frac{\sum a_k F(z_k)}{\sum a_k},$$

where $z_k = k$ are the points of grid-configuration, $a_k = A_k/(P - k)$, and $P = p + qi$ is the variable point. For example, in the case of three-point interpolation the configuration consists of the three points 0, 1, i and the values of A_k are correspondingly $A_0 = -i, A_1 = \frac{1}{2}(1 + i), A_i = \frac{1}{2}(-1 + i)$. From this we find that $\sum a_k$ reduces to $1/[P(P - 1)(P - i)]$ and the interpolation polynomial becomes

$$f(P) = -i(P - 1)(P - i)f_0 + \tfrac{1}{2}(1 + i)P(P - i)f_1 + \tfrac{1}{2}(-1 + i)P(P - 1)f_i.$$

Beginning with five-point interpolation, the author gives alternative configurations "which have the property of making the location of P (or z) considerably more central with regard to the points k (or z_k), and hence would be expected to yield greater accuracy. The number of these more central configurations which are given are: for five-point—one, for six- and seven-point—two, for eight- and nine-point—three. Thus, the user has a great latitude of choice in available formulas for complex interpolation, for checking that interpolation, and for central choice of the argument P."

This paper is a continuation of the original work of A. N. LOWAN and the author[1] [*MTAC*, v. 1, p. 358–359] and a paper by the author.[2] The work makes use of a simplification introduced by W. J. TAYLOR[3] for real Lagrangean interpolation.

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[1] A. N. LOWAN & H. E. SALZER, "Coefficients for interpolation within a square grid in the complex plane," *Jn. Math. Phys.*, v. 23, 1944, p. 156–166.

[2] H. E. SALZER, "Coefficients for complex quartic, quintic and sextic interpolation within a square grid," *Jn. Math. Phys.*, v. 27, 1948, p. 136–156.

[3] W. J. TAYLOR, "Method of Lagrangian curvilinear interpolation," NBS, *Jn. Res.*, v. 35, 1945, p. 151–155.

773[K].—D. J. GREB & J. N. BERRETTONI, "AOQL single sampling plans from a single chart and table," Amer. Stat. Assn., *Jn.* v. 44, 1949, p. 62–76.

As the title states, a single chart and table are given in the paper to find the AOQL (Average Outgoing Quality Limit) single sampling inspection plan which will yield a "practical" minimum amount of total inspection. Total inspection is used here to mean the combined amount resulting from the single sample inspected from each and every lot and the 100% screening inspection of lots rejected under the single sampling plan. Given an AOQL value in % it is desired to maintain and the lot size, N, Chart I of the paper (p. 66) is entered to find the acceptance number, c. Table II of the paper (p. 65) is then entered along with the AOQL value to find the sample size n for the single sampling plan (c, n) with $c = 0(1)12$, AOQL = .1,.25(.25)1(.5)5(1)10.

The present paper and associated tables appear to eliminate the necessity of knowing the process average accurately, at least for many practical situations.

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774[K].—D. J. Finney, "The truncated binomial distribution," *Annals of Eugenics*, v. 14, 1949, p. 319–328.

A number s of observations of binomial random variables, all with the same probability p of success, are obtained under circumstances which make it possible to observe only values greater than zero. The problem is to obtain the maximum likelihood estimate of p. Referring to his earlier paper[1] for derivations, the author presents an iterative technique for solving the likelihood equation which is made simple by giving a 3D table of the weights of single observations and of the bias in weighted scores for $s = 2(1)20$ and $p = .01(.01).05(.05).95$. A similar technique is described for the case of doubly truncated binomial distributions in which neither the number of "no successes" nor the number of "no failures" can be observed, and a similar table of weights and biases, both to 3D, for $s = 3(1)20$ and $p = .01(.01).05(.05).50$ is given.

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[1] D. J. Finney, "The estimation of the frequency of recombinations. I. Matings of known phase," *Jn. Genetics*, v. 49, 1949, p. 159–175.

775[K].—N. L. Johnson, "Systems of frequency curves generated by methods of translation," *Biometrika*, v. 36, 1949, p. 149–176.

The author investigates the properties of probability functions, p_z, where $z = \gamma + \delta \ln f(y)$ is distributed normally with unit variance, $f(y) \geqq 0$. If $f(y) = y$ we obtain the well-known log-normal system, S_L; if $f(y) = y/(1 - y)$, $0 < y < 1$, a new system S_B; and if $f(y) = y + \sqrt{y^2 + 1}$, $-\infty < y < \infty$, the new system S_U. On page 157 is a chart in terms of the Pearson measures of skewness and kurtosis, β_1 and β_2, showing the regions of S_L, S_B, and S_U. On page 164 is a nomogram which gives δ and γ/δ in terms of β_1 and β_2, $0 \leq \beta_1 \leq 1.3$, $3 \leq \beta_2 \leq 5$ for the system S_U. Table 8, page 174, gives the values of μ_1', σ, β_1, β_2 for $\delta = .5, 1, 2$, and $\gamma = 0(.5)2.5$ for S_B. The author applies his results to the graduation of observed frequency distributions and to the normalization of skewed distributions.

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776[K].—K. R. Nair, "A further note on the mean deviation from the median," *Biometrika*, v. 36, 1949, p. 234–235.

Values of the coefficient of variation (2S) are given for the mean deviation from the mean and the mean deviation from the median for samples of size 2(1)10 from a normal population. They are the same within 1 figure in the second place. This implies that the mean deviation from the median is just as precise an estimate of dispersion as the mean deviation from the mean for samples from a normal distribution.

Actually there are more precise linear estimates of dispersion than either of these. Some examples are given in a text by Dixon and the reviewer.[1]

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[1] W. J. Dixon & F. J. Massey, *Introduction to Statistical Analysis*, lithograph ed. Eugene, 1949.

777[K].—E. G. Olds, "The 5% significance levels for sums of squares of rank differences and a correction," *Annals Math. Stat.*, v. 20, 1949, p. 117–118.

In 1938 the author published[1] a series of tables having to do with the distribution of the rank correlation coefficient. Table V gave pairs of values between which $\sum d_i^2$ (d_i being the rank difference for the i^{th} individual) has a probability, P, of being included under the hypothesis that $\sum d^2 = (n^3 - n)/6$. This hypothesis is equivalent to the null hypothesis that the rank correlation coefficient is zero. The table gave pairs of values for $n = 11$ to $n = 30$ inclusive and for $P = .99, .98, .96, .90$ and $.80$. The table in the present paper extends the original table by giving pairs of 1D values corresponding to $P = .95$.

The correction is for the formula printed in the earlier publication for the variance of the normal deviate, x, used in calculating the values of the table.

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[1] E. G. Olds, "Distribution of the sums of squares of rank differences for small numbers of individuals," *Annals Math. Stat.*, v. 9, 1938, p. 133–148.

778[K].—Frances Swineford, "Further notes on differences between percentages," *Psychometrika*, v. 14, 1949, p. 183–187.

The two tables are designed to determine the least common size, N, for each of two samples for testing the hypothesis that the difference in the two population proportions, $p_1 - p_2$, is at least d_t. It is assumed that the sample proportions, p_1' and p_2', are distributed normally and, therefore, that the appropriate test is a one-tailed test of the hypothesis that $p_1 - p_2 = d_t$. Then $N = 5.4119(p_1q_1 + p_2q_2)/(d_0 - d_t)^2$ at the 1% point and approximately half as much (1/2.0003) at the 5% point, where $d_0 = p_1' - p_2'$. The tables give N for the 1% points only.

Table 1 gives $N' = 10.8238pq/(d_0 - d_t)^2$, where $p = \frac{1}{2}(p_1' + p_2')$, to 0D for $p = .10(.05).90$ and $|d_0 - d_t| = .050(.002).080(.005).135$. Table 2 gives the correction terms $(p_1q_1 + p_2q_2)/2pq$ to 3D(3S) for $p = .10(.05).90$ and $d_t = .10(.05).50$.

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779[K].—J. E. Walsh, "On the power function of the 'best' t-test solution of the Behrens-Fisher problem," *Annals Math. Stat.*, v. 20, 1949, p. 616–618.

Let m sample values be drawn from $N(a_1, \sigma_1^2)$ and n sample values from $N(a_2, \sigma_2^2)$, $m \leq n$, (where $N(a, \sigma^2)$ represents a normal probability function with mean a and variance σ^2), then Scheffé has shown that a "t" test solution of the Behrens-Fisher problem, σ_1^2/σ_2^2 not known, using in the numerator the difference of sample means, and the denominator based on the square root of a function of the sample values which has a χ^2-distribution with $m - 1$ degrees of freedom has certain optimum properties. The purpose

of the note is to compare the power function of this t test with the power function of the correspondingly most powerful test for the case in which the ratio σ_1^2/σ_2^2 is known, for one-sided and two-sided symmetrical tests. This is done by finding the power efficiency. Two 3D tables are given for the power efficiency of Scheffé's test, σ_1^2/σ_2^2 not known, against the test σ_1^2/σ_2^2 known, on page 617 where α = significance level, for $\alpha = .05$, $m = 4, 6, 10, 15, 20, 30, 50, 100, \infty$ and the same range for n; also $\alpha = .01$, $m = 6, 8, 10, 15, 20, 30, 50, 100, \infty$, same range for n.

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780[K].—J. E. Walsh, "On the 'information' lost by using a t-test when the population variance is known," Amer. Stat. Assn., *Jn.*, v. 44, 1949, p. 122–125.

Table 1 gives the approximate number of sample values "wasted" if, when the population variance is known, one uses a t-test (estimating variance from sample) in place of the appropriate normal deviate test, when one is testing whether the population mean differs from a given constant value. 5%, 2.5%, 1% and .5% significance levels are tabulated to 2S for both the one-sided and symmetrical test. Sample values wasted is defined in terms of equal power functions.

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781[K].—J. E. Walsh, "Applications of some significance tests for the median which are valid under very general conditions," Amer. Stat. Assn., *Jn.*, v. 44, 1949, p. 342–355.

The tests considered are valid for samples of n from one or more universes with a common median, which are symmetric and have continuous cumulative distributions. Table I lists one to five such tests for each $n = 4(1)15$ with their approximate one-sided and symmetric significance levels to 3D and their efficiencies to nearest .5 on the assumption that the universes sampled are normal. Table II lists further tests for $n = 4(1)9$ for which the bounds of these significance levels are given, as well as their significance levels and efficiencies to same precision as Table I for samples from normal. Table I was also published elsewhere.[1]

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[1] J. E. Walsh, "Some significance tests for the median which are valid under very general conditions," *Annals Math. Stat.*, v. 20, 1949, p. 64–81.

782[L].—S. Chandrasekhar, "On Heisenberg's elementary theory of turbulence," R. Soc. London, *Proc.*, v. 200A, 1949, p. 20–23.

The function $f(x)$ satisfies a certain nonlinear differential equation. Introducing the new variables

$$y = \int_0^x f(t)t^2dt \quad \text{and} \quad g = x^3f(x),$$

and denoting differentiation with respect to y by primes, the differential equation becomes

$$g^{\frac{1}{2}}g'' + 2y(4 + g') + 2g^{\frac{1}{2}}(4 - g') - 8g = 0.$$

For small y,

$$g(y) = 4y + y^{\frac{1}{2}}(a + (4/3)\ln y) + \cdots$$

where a is an arbitrary constant. Starting with this value, the differential equation was integrated numerically for $a = 1.8104739, 1.81, 1.75, 1.5, 1.0, .5, 0$. Four decimal tables of $f(x)$ are given for various ranges of x and varying intervals.

A. E.

783[L].—L. HOWARTH, "Rayleigh's problem for a semi-infinite plate," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 127–140.

The work leads to the expression

$$\tau = \frac{\mu W}{(\nu t)^{\frac{1}{2}}}\left\{\frac{1}{\pi^{\frac{1}{2}}} + \frac{1}{(2\pi)^{\frac{1}{2}}}\int_R^\infty e^{-(1/8)v^2}K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\frac{dv}{v}\right\},$$

and the notation $\tau_R = (\pi\nu t)^{-\frac{1}{2}}\mu W$ is used. Table 1 gives values to 3 decimal places of $(W^2t/\nu)^{\frac{1}{2}}\tau/(\rho W^2)$ and of $(\mu W)^{-1}\int_0^r(\tau - \tau_R)dr$ for $R = (\nu t)^{-\frac{1}{2}}r = .01, .05, .1, .2(.2)2(.5)4$ and ∞.

A. E.

784[L].—M. K. KROGDAHL, "On broadening of hydrogen lines in stellar spectra—I," *Astrophys. Jn.*, v. 110, 1949, p. 355–374.

The function

$$-\frac{1}{15}\left(\frac{z}{2}\right)^4\sin\frac{2}{z} + \frac{1}{15}\left(\frac{z}{2}\right)^3\cos\frac{2}{z} - \frac{29}{30}\left(\frac{z}{2}\right)^2\sin\frac{2}{z} - \frac{1}{6}\frac{z}{2}\cos\frac{2}{z}$$
$$+\frac{4}{3}\frac{z}{2}\int_{2/z}^\infty\cos y\frac{dy}{y} + \frac{1}{6}\int_{2/z}^\infty\sin y\frac{dy}{y} + \frac{32}{15}\left(\frac{z}{2}\right)^{\frac{1}{2}}\int_0^{2/z}\cos y\frac{dy}{\sqrt{y}}$$

is tabulated to 3D for $z = 0(.1)1(.2)3(.5)5$, and to 2D for $z = 6(1)10$. The computations were apparently based on tables contained in JAHNKE-EMDE. [1933 ed. p. 6–9, 34–35].

A. E.

785[L].—MERBT, "Untersuchung zur Arbeit von H. G. Küssner 'Lösungen der klassischen Wellengleichung für bewegte Quellen'" Aerodynamische Versuchsanstalt, Göttingen e. V., Abteilung J 06, *Bericht* B 44/J/31, ZWB/AVA/Re/44/J/31, ZWB 10514, 1944. English translation "Wave propagation from moving sources" by O. W. Leibiger Research Laboratories, Petersburg, N. Y., ATI No. 32413, 8–8–701, 1948. 14 p.

The function

$$h_{mn}(\alpha, \sigma) = \frac{1}{2\pi}\int_0^{2\pi}(1 - 2\sigma\cos\chi)^{-\frac{1}{2}}\exp\{imn[-\chi - \alpha(1 - 2\sigma\cos\chi)^{\frac{1}{2}}]\}d\chi$$

occurs in an investigation[1] of sound propagation from a source moving in a circle. In the present paper the following notations are used:

$$\beta = imn\alpha, \qquad A = (1 - 2\sigma \cos \chi)^{\frac{1}{2}}, \qquad y = e^{\beta\sigma\cos\chi},$$
$$F(y) = A^{-1}e^{-\beta A}, \qquad f(\sigma) = e^{-\beta A}.$$

First the first 8 derivatives of $F(y)$ are given in terms of β and A, then the values, in terms of β, of these derivatives when $y = 1$. From these the first 7 coefficients in the asymptotic expansion

$$h_{mn}(\alpha, \sigma) = i^{mn} \sum_{\nu=1}^{\infty} b_\nu J_{mn}(\nu mn\alpha\sigma)$$

are computed. Next, the first 6 derivatives of $f(\sigma)$ are given in terms of β and A, and the values of these derivatives for $\sigma = 0$ in terms of β and χ. The Maclaurin series of $f(\sigma)$ leads to an expansion of h in powers of σ which is said to show satisfactory convergence, at any rate for small m, for all values of α and σ which are of importance; in fact it is stated that a few terms of the series suffice for numerical computation. Certain polynomials which occur in the first 7 coefficients of the expansion of h_{mn} in powers of σ are given explicitly. There are 14 such polynomials, 7 to be used for even m and 7 for odd m. Table I of the appendix gives the coefficients of these polynomials, and Table II certain auxiliary quantities which can be used for a rapid computation of the coefficients.

A. E.

[1] H. G. KÜSSNER, "Lösungen der klassischen Wellengleichung für bewegte Quellen," *Z. angew. Math. Mech.*, v. 24, 1944, p. 243–250.

786[L].—F. W. J. OLVER, "Transformation of certain series occurring in aerodynamic interference calculations," *Quart. Jn. Mech. Appl. Math.*, v. 2, 1949, p. 452–457.

$$k_s = 8\pi\mu^2 s \sum_{n=1}^{\infty} (-1)^{n-1} K_0(2\pi\mu sn) + 4\mu \sum_{n=1}^{\infty} (-1)^{n-1} \frac{K_1(2\pi\mu sn)}{n}$$

is given in Table 1 (p. 456) to 6 decimal places for $\mu = .5$, $s = 1(1)6$; $\mu = 1$, $s = 1, 2, 3$; and $\mu = 2$, $s = 1, 2$.

Table 2 (p. 457) is an auxiliary table giving values to various degrees of accuracy of $K_j(n\pi)$ and $I_j(n\pi)$ for $j = 0, 1$ and $n = 1(1)7$.

787[L].—F. RIEGELS, "Formeln und Tabellen für ein in der räumlichen Potentialtheorie auftretendes elliptisches Integral," *Archiv d. Math.*, v. 2, 1949–50, p. 117–125.

The integral in question is

$$G_n(k^2) = (-1)^n \int_0^{\frac{1}{2}\pi} \frac{\cos 2n\theta}{(1 - k^2 \sin^2 \theta)^{\frac{3}{2}}} d\theta,$$

and can be expressed in terms of complete elliptic integrals of the first and second kinds in the form

$$(1 - k^2)^{-1} k^{-2n} [f_n E(k^2) - g_n K(k^2)],$$

where f_n and g_n are polynomials of degree n in k'^2. The exact values of the coefficients of these polynomials are given (as common fractions) for $n = 0(1)7$.

$(2/\pi)G_n(k^2)$ can be expanded as a power series in k^2, and 5 decimal values of the coefficients of this power series up to the coefficient of k^{30} are given for $n = 0(1)7$.

$k'^2G_n(k^2)$ can also be expanded in the form: power series in $k'^2 + \ln(4/k')$ times a power series in k'^2. The coefficients occurring in this expansion up to (and including) the terms in k'^{10} are tabulated to 5 or 6 significant places for $n = 0(1)7$.

The principal tables are 4 decimal tables of k'^2 and of $k'^2G_n(k^2)$ for $n = 0(1)3$, $k^2 = .00(.01).99$ and $k^2 = .900(.001).999$.

The computations were carried out on the Hollerith equipment of the Max Planck Gesellschaft of Göttingen. The original computations used the interval .001 throughout and included also G_4 to G_7. The unpublished parts of the tables are available through the author.

A. E.

788[L].—G. Schweikert, "Zur Theorie des Gasdrucks gegen eine bewegte Wand," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 289–300.

Let

$$F(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty \exp(-t^2)dt$$

and

$$G(x) = \pi^{\frac{1}{2}}x \exp(x^2)F(x).$$

The article contains tables to 5D of

$$F(x), \quad 2x^2F(x), \quad (2 - F)(1 + 2x^2) + 2\pi^{-\frac{1}{2}}x \exp(-x^2)$$

and

$$[1 + 2x^2(1 - \theta^2)]F(x) - 2\pi^{-\frac{1}{2}}x \exp(-x^2)$$

for

$$\theta = 0(.1).4, 1 \text{ and for } x = 0(.1)1.5, 2, 2.5$$

and tables to 8D of $\log G(x)$, $1/G(x)$ and

$$1 + \tfrac{1}{2}x^{-2} - 1/G(x)$$

for $x = .05, .1(.1)2(.5)4.5$.

789[L].—Geoffrey Taylor, "The formation of a blast wave by a very intense explosion. I. Theoretical discussion." R. Soc. London, *Proc.*, v. 201A, 1950, p. 159–174. "II. The atomic explosion of 1945," *ibid*, p. 175–186.

In the course of this work there appear the functions f, ϕ, ψ of η which satisfy the system of nonlinear differential equations

$$(\eta - \phi)\phi' = \frac{1}{\gamma}\frac{f'}{\psi} - \tfrac{3}{2}\phi, \qquad \frac{\psi'}{\psi} = \frac{\phi' + 2\phi/\eta}{\eta - \phi}$$

$$3f + \eta f' + \frac{\gamma\psi'}{\psi}(-\eta + \phi)f - \phi f' = 0$$

in which γ is a constant.

Table 1 of Part I (p. 164) contains 3 decimal values of f, ϕ, ψ for $\eta = 1(-.02).5$. These values were computed by step-by-step numerical integration of the differential equations with $\gamma = 1.4$ and the initial values $f(1) = 1.167$, $\phi(1) = .833$, $\psi(1) = 6.000$.

Approximate formulas for the functions in question are developed on p. 165.

Table 2 of Part I (p. 166) contains 3 decimal values of f, ϕ and 2 decimal values of ψ for $\eta = 1, .95, .9, .8, .7, .5, 0$. These values were computed from the approximate formulas with $\gamma = 1.666$, $f(1) = 1.250$, $\phi(1) = .750$, $\psi(1) = 4.000$.

Table 2 of Part II (p. 178) contains 3 decimal values of f, ϕ, ψ for $\eta = 1(-.02).9(-.05).4$ obtained by approximate calculation for $\gamma = 1.30$, $f(1) = 1.130$, $\phi(1) = .869$, $\psi(1) = 7.667$. Some values of the temperature are added.

There are also other tables.

A. E.

790[L].—A. VAN WIJNGAARDEN & W. L. SCHEEN, "Table of Fresnel integrals," Akademie van Wetenschappen, Amsterdam, *Afd. Natuurkunde, Verhandelingen*, eerste sectie, v. 19, no. 4, 1949, 26 p. (Report R49 of the Computation Department of the Mathematical Centre at Amsterdam.) Price 2.50 guilders.

This is a table of

$$C(u) = \int_0^u \cos \tfrac{1}{2}\pi t^2 dt \quad \text{and} \quad S(u) = \int_0^u \sin \tfrac{1}{2}\pi t^2 dt$$

for $u = [0(.01)20; 5D]$ with modified second differences.

$$(1) \quad C(u) = \sum_{k=0}^{\infty} C_{4k+1} u^{4k+1}, \qquad C_{4k+1} = (-1)^k(\tfrac{1}{2}\pi)^{2k}/[(2k)!(4k+1)]$$

$$(2) \quad S(u) = \sum_{k=0}^{\infty} S_{4k+3} u^{4k+3}, \qquad S_{4k+3} = (-1)^k(\tfrac{1}{2}\pi)^{2k+1}/[(2k+1)!(4k+3)].$$

Tables are given for $k = 0(1)22$ or 21, of $C_{4k+1}[C_1 = 1, C_5 = -2.467401100272340 \cdot 10^{-1}, \cdots, C_{89} = 2 \cdot 10^{-49}]$, and of $S_{4k+3}[S_3 = 5.235987755982989 \cdot 10^{-1}, \cdots, S_{87} = -5 \times 10^{-47}]$.

Asymptotic expansions are

$$(3) \quad C(u) \sim \tfrac{1}{2} + \sin \tfrac{1}{2}\pi u^2 \sum_{k=0}^{\infty} \gamma_{4k+1} u^{-(4k+1)} - \cos \tfrac{1}{2}\pi u^2 \sum_{k=0}^{\infty} \sigma_{4k+3} u^{-(4k+3)}$$

$$(4) \quad S(u) \sim \tfrac{1}{2} - \cos \tfrac{1}{2}\pi u^2 \sum_{k=0}^{\infty} \gamma_{4k+1} u^{-(4k+1)} - \sin \tfrac{1}{2}\pi u^2 \sum_{k=0}^{\infty} \sigma_{4k+3} u^{-(4k+3)}$$

where

$$(5) \quad \gamma_{4k+1} = (-1)^k 2(4k)!(2\pi)^{-(2k+1)}/(2k)!$$

$$(6) \quad \sigma_{4k+3} = (-1)^k 2(4k+2)!(2\pi)^{-(2k+2)}/(2k+1)!$$

There are tables of the coefficients (5) and (6) for $k = 0(1)14$ and $k = 0(1)13$ respectively.

In computing the main table 9D values were calculated by means of (1) and (2) for $u = 0(.5)2.5$ and the same from (3), (4) for $u = 2.5(.5)12$. In order to get this accuracy by means of asymptotic series for values of u as low as 2.5, the technique described by GOODWIN & STATON[1] was employed. The next step was to prepare preliminary 7D tables of $C(u)$ and $S(u)$ for $u = 0(.01)12$ by numerical integration of 5D values of $\cos \frac{1}{2}\pi t^2$ and $\sin \frac{1}{2}\pi t^2$ with an interval $h = .01$. After most elaborate treatments and testings, the authors were led to their rounded-off 5D values, which are "guaranteed."

The values of the table for $u = 12(.01)20$ were calculated by means of the asymptotic series and checked by complete duplication of the computation. Modified second differences, $\delta^{2*} = \delta^2 - 0.184\delta^4$ were computed from the 7D tables, then rounded off to 5D and checked by differencing.

Linear interpolation will yield no larger error than $4 \times 10^{-6} \times u$. Full profit of the accuracy of the table is obtained by the use of EVERETT's formula up to second modified differences:

$$f(x_0 + ph) = (1 - p)f_0 + pf_1 + E_0{}^2\delta_0{}^{2*} + E_1{}^2\delta_1{}^{2*}.$$

A small table of the interpolation polynomials $E_0{}^2$ and $E_1{}^2$ is given.

Extracts from text.

We have recently referred in *MTAC* to several other tables of $C(u)$ and $S(u)$: C. M. Sparrow (v. 3, p. 479), for $u = 0(.005)8$; 4D; D. L. ARENBURG & D. LEVIN (v. 3, p. 479), for $u = 0(.1)20$, and $u = 8(.02)16$; U. S. NAVY, RES. LAB., Boston (v. 3, p. 417), for $u = 0(.1)20$; 4D or 4S; R. T. BIRGE (v. 4, p. 30), for $u = 0(.05)12.05$; 4D.

A comparison of the first 100 values of $C(u)$ and $S(u)$ of tables under review with the corresponding entries in the Sparrow tables indicated the following 29 apparent unit-errors in the fourth places of Sparrow: $S(.15)$, $C(.23)$, $S(.31)$, $C(.33)$, $S(.33)$, $C(.45)$, $C(.47)$, $C(.49)$, $S(.57)$, $S(.59)$, $S(.63)$, $S(.65)$, $S(.69)$, $C(.72)$, $C(.73)$, $C(.74)$, $C(.75)$, $C(.76)$, $S(.77)$, $C(.79)$, $S(.79)$, $C(.84)$, $S(.85)$, $C(.91)$, $C(.93)$, $S(.93)$, $S(.94)$, $C(.95)$, $C(.98)$. There are also 6 2-unit errors at $C(.77)$, $C(.78)$, $C(.80)$, $C(.81)$, $C(.82)$, $C(.83)$. These results suggest that tables based on Sparrow are likely to be not without error.

Wijngaarden & Scheen make no reference to any earlier table of Fresnel integrals. A misprint for $C(4.95)$, 0.45404 has been corrected by hand to 0.54504.

R. C. A.

[1] E. T. GOODWIN & J. STATON, "Table of $\int_0^\infty e^{-u^2}du/(u + x)$," *Quart. Jn. Mech. Appl. Math.*, Oxford, v. 1, 1948, p. 319–326. See *MTAC*, v. 3, p. 483.

791[L].—D. V. ZAGREBIN, "K voprosu o tochnosti formuly Stoksa" [Concerning the accuracy of Stokes' formula], Akad. Nauk. SSSR, Inst. Teoret. Astr., *Bull.*, v. 4, no. 3 (56), 1949, p. 134–141.

Table 1, p. 137, is a 3D table of six very special functions which are combinations of trigonometric and logarithmic functions with complete elliptic integrals. There are also tables to 3,4D of 15 definite integrals of products of these functions by even powers of the sine and cosine.

792[M].—Roy C. Spencer, Pauline Austin, Elizabeth Chisholm, Ellen Fine, & Jeane Schwartz, *Tables of Fourier Transforms of Fourier Series, Power Series, and Polynomials. Report S-58, July 10, 1945.* Radiation Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., ii, 29 p. 21.4 × 27.9 cm.

The tables, occupying p. 6–27, are as follows:

T. I, p. 6–11, Fourier transform of a constant, $g_0(\phi) = \phi^{-1}\sin\phi$, g_0^2, its derivative Dg_0, $(Dg_0)^2$, $\phi = [0(5°)1080°; 8D]$.
T. II, p. 12–13, $D^n g_0$ for $n = 1(1)8$, $\phi = [0(30°)1080°; 8D]$.
T. III, p. 14, Fourier transforms T of $\cos n\pi x$ and $\sin n\pi x$ for $n = 1(1)3$. Tables of $T\cos n\pi x$, $iT\sin n\pi x$, $n = 1(1)3$, $\phi = [0(30°)720°; 7D]$.
T. IV, p. 15, Fourier transforms of $\cos\frac{1}{2}n\pi x$, $\sin\frac{1}{2}n\pi x$, $n = 1, 3$. Tables of $T\cos\frac{1}{2}n\pi x$, $iT\sin\frac{1}{2}n\pi x$, $n = 1, 3$, $\phi = [0(30°)720°; 7D]$.
T. V, p. 16, $T(1 - x^2)^n$, $n = 1(1)3$, $\phi = [0(30°)720°; 7D]$.
T. VI, p. 17, Fourier transforms, $TP_n(x) = i^n j_n(\phi)$, $n = 2(1)4$. Tables of $j_n(\phi)$, $n = 2(1)4$, $\phi = [0(30°)720°; 7D]$.
T. VII, p. 18–25, Fourier transforms of $i^n x^n$, g_0, $D^n g_0$, $n = 1(1)8$, $\phi = [0(.1)20; 8D]$.
T. VIII, p. 26–27, Fourier transforms of $(1 - x^2)^n$, $(1 + D^2)^n g_0$, $n = 1(1)4$, $\phi = [0(.1)10; 8D]$.

There are a number of disagreements with values given in the tables of RMT 726 [*MTAC*, v. 4, p. 80–81].

R. C. A.

793[U].—Pierre Hugon, *Nouvelles Tables pour le calcul de la droite de hauteur à partir du point estime*, Paris, Girard, Barrère and Thomais, 1947, xiv, 92 p. 15.0 × 21.2 cm. + 1 chart 25 × 42.4 cm.

The tables are two in number; the principal table was designed for use on shipboard in calculating the altitude for the dead reckoning position by logarithms and haversines. It consists of 90 pages of five-place values of the natural haversine, log haversine, and log cohaversine, with argument 1°(1′)179°; each page contains an interpolation table for each of the three functions for tenths of a minute of arc. Preceding the principal table are two pages of corrections for refraction, height of eye and semidiameter to be applied to observed altitudes of the lower limb of the sun and of stars. The arguments are observed altitude 6°(1°)16°(2°)20°(5°)50°(10°)90° and height of eye 3(1)10(2)26 meters.

The foreword and explanation are presented first in French and again in English. The formulas used are:

$$\begin{aligned}\text{hav}\,(90° - h) &= A + B\\ \log A &= \log\text{cohav}\,t + \log\text{hav}\,(d - L)\\ \log B &= \log\text{hav}\,t + \log\text{cohav}\,(d + L)\end{aligned}$$

where t, d, and h are the local hour angle, declination and altitude of the celestial body and L is the dead reckoning latitude. Since the haversine and the cohaversine are always positive and between zero and one, the rule of signs is no longer needed. Also since corresponding values of the natural haversine and log haversine are given side by side, no separate table of logarithms of numbers is given.

The azimuth is determined by the use of a nomogram folded inside the back cover; it is based on the formula:

$$\cos h \cos Z = \sin (d - L) \operatorname{cohav} t + \sin (d + L) \operatorname{hav} t$$

where Z is the azimuth angle of the celestial body. It is intended that the azimuth shall be determined only to the nearest degree which is generally adequate for ordinary navigational purposes.

The author suggests that the accuracy of his method as compared to the classical FRIOCOURT method is as follows:

	$h = 60°$	$h = 75°$	$h = 84°$
Friocourt	$+0'.5$	$+0'.9$	$+2'.2$
Hugon	$+0'.4$	$+0'.7$	$+1'.7$

and hence the claim is for greater speed and ease of use rather than greater accuracy with the same number of decimals.

The printing of the tables is rather poor on the whole, but a part of the trouble may be blamed on the quality of the paper which is mediocre. It is to be hoped that the proofreading of the tables has been done with greater care than that of the foreword and explanation. In the English explanation, $X - Y = 1$ should obviously be $X + Y = 1$, and in the expression for X in both the French and English explanations, cohaversine $(D + \phi)$ should be cohaversine P.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to errata in RMT **790** (Wijngaarden & Scheen), **792** (Spencer *et al.*), **793** (Hugon), and Note **118**.

173.—*Polnoe Sobranie Sochinenii P. L. Chebysheva* [Complete Collection of Works by P. L. Chebyshev]. Volume 1, *Teoriia Chisel* [Theory of Numbers], Moscow and Leningrad, Academy of Sciences, 1946. 342 p. + portrait frontispiece. 15 × 23 cm. 20 roubles paper; 23 roubles bound. Edition (Second, stereotyped) of 3,000 copies.

Previous editions of this volume have been reviewed in *MTAC*, v. 1, p. 440–441. The present volume not only reproduces the errata of the 1944 edition but adds many new misprints both in the text and in the tables, p. 311–339. These latter are as follows:

page		
311		line −20 *for* 2372 *read* 2237
314,	$p = 13$,	$N = 12$ *read* $I = 6$
	$p = 19$,	*insert* $N = 1$
	$p = 23$,	*insert* 4 between 17 and 5
317,	$p = 61$,	N table, *for* line 1 *read* 1 10 39 24 57 21 27 26 16 38
	$p = 67$,	$I = 47$ *for* $N = 38$ *read* $N = 18$
318,	$p = 71$,	$N = 16$ *for* $I = 15$ *read* $I = 22$
		$N = 26$ *for* $I = 22$ *read* $I = 15$
319,	$p = 89$,	insert the primitive root 35

page				
320, $p = 103$, $I = 99$	*read*	$N = 31$		
322, $p = 127$, $I = 116$	*for*	$N = 31$	*read*	$N = 71$
323, $p = 131$, $I = 35$	*for*	$N = 76$	*read*	$N = 79$
324, $p = 139$, $N = 50$	*for*	$I = 26$	*read*	$I = 25$
$I = 132$	*for*	$N = 47$	*read*	$N = 57$
325, $p = 151$, $N = 12$	*for*	$I = 121$	*read*	$I = 131$
$I = 138$	*for*	$N = 81$	*read*	$N = 91$
$p = 157$, $N = 36$	*for*	$I = 80$	*read*	$I = 70$
326, $p = 163$, $N = 143$	*for*	$I = 134$	*read*	$I = 154$
$p = 167$, $N = 109$	*for*	$I = 36$	*read*	$I = 35$
$N = 113$	*for*	$I = 193$	*read*	$I = 103$
$N = 161$	*for*	$I = 144$	*read*	$I = 147$
$I = 147$	*for*	$N = 61$	*read*	$N = 161$
$I = 162$	*for*	$N = 21$	*read*	$N = 25$
327, $p = 173$, $N = 21$	*for*	$I = 138$	*read*	$I = 38$
$N = 49$	*for*	$I = 61$	*read*	$I = 62$
$N = 57$	*for*	$I = 72$	*read*	$I = 92$
328, $p = 179$, $I = 79$	*for*	$N = 36$	*read*	$N = 33$
$p = 181$, $N = 16$	*for*	$I = 175$	*read*	$I = 172$
$N = 99$	*for*	$I = 192$	*read*	$I = 102$
$N = 102$	*for*	$I = 79$	*read*	$I = 76$
329, $p = 191$, $N = 91$	*for*	$I = 9$	*read*	$I = 99$
$I = 0$	*for*	$N = 0$	*read*	$N = 1$
$I = 172$	*for*	$N = 1\ 8$	*read*	$N = 138$
$p = 193$, $N = 42$	*for*	$I = 133$	*read*	$I = 138$
330, $p = 193$, $I = 115$	*for*	$N = 182$	*read*	$N = 82$
331, $p = 199$, $N = 11$	*for*	$I = 89$	*read*	$I = 189$
$N = 56$	*for*	$I = 10$	*read*	$I = 20$
$I = 32$	*for*	$N = 12$	*read*	$N = 7$

All the errata in the table of linear divisors of quadratic forms listed in *MTAC*, v. 1, p. 441 as appearing in the 1944 edition are present in this new edition with the addition of the following new errata:

page	form				
332	$x^2 + 13y^2$	*for*	$2z + 1$	*read*	$52z + 1$
333	$x^2 + 46y^2$		*insert* 167		
	$x^2 + 69y^2$	*for*	77	*read*	73
	$x^2 + 71y^2$	*for*	2 7	*read*	237
334	$x^2 + 87y^2$	*for*	17	*read*	317
	$x^2 + 89y^2$	*for*	106	*read*	105
335	$x^2 + 93y^2$		*insert* 121		
	$x^2 + 95y^2$	*for*	36	*read*	363
336	$x^2 - 29y^2$		*insert* 83		
	$x^2 - 34y^2$		*omit* 117		
	$x^2 - 41y^2$	*for*	63	*read*	73
	$x^2 - 43y^2$	*for*	7	*read*	97
	$x^2 - 47y^2$		*insert* 65		
		for	2 7	*read*	187

page					
336	$x^2 - 51y^2$		*insert* 175		
	$x^2 - 53y^2$	*for*	1 1	*read*	131
	$x^2 - 55y^2$		*insert* 67 *and* 201		
	$x^2 - 58y^2$	*for*	67	*read*	65
337	$x^2 - 74y^2$		*insert* 253		
338	$x^2 - 85y^2$	*for*	73	*read*	173
339	$x^2 - 101y^2$	*for*	378	*read*	373

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UNPUBLISHED MATHEMATICAL TABLES

Editorial Note: The UMT File [see *MTAC*, v. 4, p. 101] now contains the following manuscript table: UMT 77[D].—E. C. Bower, *Natural Circular Functions for decimals of a circle* [*MTAC*, v. 3, p. 425]. For an unpublished table concerning elliptic integrals see RMT 787.

95[A].—Institut für angewandte Mathematik, Eidg. Tech. Hochschule, Zürich, *Table of Binomial Coefficients*. Manuscript in the possession of the Institute.

This is a table of the exact values of the binomial coefficients, extending as far as

$$\binom{100}{50} = 10089\ 13445\ 45564\ 19333\ 48124\ 97256.$$

H. Rutishauser

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96[A].—J. W. Wrench, Jr., & L. B. Smith, *Values of the terms of the Gregory series for* arccot *5 and* arccot *239 to 1120 and 1150 decimal places, respectively*. Mss. in possession of the authors.

The table of individual terms of the Gregory series for arccot 5 gives, in the range 501D to 1150D inclusive, the first 820 terms of that series. Exclusive of zeros following terminating decimals, the total number of significant figures involved is 379,290. The companion table of terms of the series for arccot 239 consists of 1120D values of the first 235 terms. The total number of significant figures in this table is 131442.

The sums of the positive and negative terms of each series are given to the corresponding degree of approximation. From these data approximations to arccot 5 and arccot 239 have been obtained correct to 1148D and 1119D, respectively, as confirmed by the ENIAC calculation of these numbers [*MTAC*, v. 4, pp. 11–15].

For the sake of chronological accuracy it should be mentioned that the final checking of the 1120D approximation to arccot 239 was completed by Mr. Smith on 24 July 1949, and the calculation of arccot 5 had been completed by the writer the previous month except for checking the data beyond

850D. This checking had not been completed when the ENIAC computation was made in September, 1949, and subsequent comparison with these independently computed values of arccot 5 and arccot 239 revealed several discrepant figures which were found to be due entirely to seven errors of transcription of data and addition of terms in the previously unchecked portion of the calculation of arccot 5. Mr. Smith's value of arccot 239 agreed perfectly to 1119D with the more extended approximation found by the ENIAC. By 6 October 1949 all discrepancies had been removed and the derived approximation to π agreed through 1118D with the value determined by the ENIAC.

As a by-product of the extension of arccot 5 from 850D to 1150D the earlier table of 2^n, $n = 1(2)1207$ [*MTAC*, vol. 2, pp. 246, 374], has been extended to the range $n = 1(2)1667$. The same method of checking that was employed before, namely Fermat's simple theorem, was retained. In addition, the last entry in the table was multiplied by 2^{555} to obtain 2^{2222}, which had previously been computed by H. S. UHLER but not published. The two values of this power agreed perfectly.

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Washington 16, D. C.

97[C].—RADIO CORPORATION OF AMERICA, *Table of Logarithms to the Base 2 for numbers* 0.000 *to* 1.000, Mimeograph MS in possession of the RCA Laboratories Division, Princeton, N. J. Copy deposited in the UMT FILE.

The function $\log_2 p$ is tabulated to 4D for

$$p = 0(.002).2(.01)1.$$

The table was prepared to facilitate studies in the theory of information-handling.

R. SERRELL

RCA Laboratories
Princeton, N. J.

98[D, H].—SIDNEY JOHNSTON. *Solutions of* $\sin x = \pm cx$. Manuscript in possession of the author, 81 Fountain St., Manchester 2, England.

This manuscript gives a solution x of $\sin x = cx$ for $c = 0(.002).78(.001).927(.0005).9665(.00025)98575,9854(.0002).9892(.0001).9951(.00005).9979$ and of $\sin x = -cx$ for $c = 0(.001).136(.0005).1815(.00025).201(.0002).2054(.0001).2122(.00005).2148(.00002).217$ and a large number of values of c in the immediate neighborhood of $c = .217$. There is also a table of solutions of $\sin x = (1 - c^2)x$ for $c = 0(.002).048$. Throughout the entire table x is given to 8D with Δ, Δ^2, Δ^3, Δ^4.

99[D, H].—SIDNEY JOHNSTON. *Solutions of* $x - \sin x = c$. Manuscript in possession of the author and NBSCL.

This gives solutions to 8D of $x - \sin x = y^3/6$ for $y = 0(.01)1.53$.

100[F].—R. C. Douthitt, *Tables related to Euler's totient function*. On punch cards in possession of the author, Dept. of Math., University of California, Berkeley. Tabulated manuscript in UMT File.

We denote as usual by $\phi(n)$ the number of numbers not exceeding n and prime to n and define

$$\Phi(n) = \sum_{m\leq n} \phi(m)$$
$$A(n) = [\tfrac{1}{2} + 3n^2\pi^{-2}]$$
$$B(n) = \Phi(n) - A(n)$$
$$C(n) = A(n) - \Phi(n-1).$$

The table gives all five functions for $n = 1(1)10000$. The values of $\phi(n)$ were taken from Glaisher's tables [*MTAC*, v. 1, p. 136], and the other functions were computed and checked by means of IBM 604 and 602A calculating punch machines.

The functions $B(n)$ and $C(n)$, contrary to a conjecture by J. J. Sylvester, are not always positive and are found to be negative or zero for a total of 20 times for $B(n)$ and 18 times for $C(n)$. [See also UMT 86[F], *MTAC*, v. 4, p. 29–30.]

101[F].—D. H. Lehmer, *Tables of Ramanujan's* $\tau(n)$. Tabulated manuscript and punched cards deposited in UMT File.

The function $\tau(n)$ is the coefficient of x^{n-1} in the expansion of the 24-th power of Euler's infinite product

$$(1-x)(1-x^2)(1-x^3)\cdots.$$

The tables give $\tau(n)$ for $n = 1(1)2500$ as well as

$$\sum_{n\leq N} \tau(n), \qquad \sum_{n\leq N} |\tau(n)|, \qquad \sum_{n\leq N} \{\tau(n)\}^2$$

for $N = 10(10)2500$. There is a separate table of $\tau(p)$ for $1000 < p < 2500$ and p a prime, giving also 6D values of $|\tau(p)|p^{-11/2}$. The table was produced on an IBM 602A calculating punch.

102[L].—Cambridge University Mathematical Laboratory. *Table of* $1/\Gamma(x+iy)$. Manuscript in the possession of the Laboratory.

The table furnishes real and imaginary parts of the reciprocal of the Gamma function, $1/\Gamma(x+iy)$, to 6D for $x = -.5(.01).5$ and $y = 0(.01)1$. This table was computed and printed by the EDSAC (*MTAC*, v. 4, p. 61–65) under the direction of J. P. Stanley and involved nearly 20 hours of machine time.

103[L].—SIDNEY JOHNSTON, *Tables of Sievert's Integral*, manuscript in possession of the author, photograph copy at NBSCL.

This is a table of the function

$$\int_0^x \exp(-A \sec t)dt$$

for $A = 0(.5)10$ and $x = 0(\pi/180)\pi/2$. Values are given mostly to 5S. Explicit tabulation is not made beyond a value of x where the integral remains unchanged to 5S.

104[V].—SIDNEY KAPLAN, *Tables of Velocity Functions Characterizing Flows Formed by Jets from Orifices*. U. S. Naval Ordnance Laboratory Memorandum, 87 p. Available only to government agencies and contractors.

The basic mathematics governing the flow of incompressible fluids has been known for many years. Because a great amount of tedious computation is necessary, flows for only a few isolated cases have been calculated in the past. At the suggestion of G. BIRKHOFF,[1] the Naval Ordnance Laboratory has calculated for the first time the flow patterns of an incompressible fluid from an orifice for four different angles of aperture: $\alpha = 0°, 15°, 45°$, and $90°$. In all, more than 2,000 points were calculated.

The governing equations are

$$(1) \qquad W = U + iV = \ln(\zeta) - \ln(\zeta^2 - 2C\zeta + 1)$$

$$(2) \qquad Z = x + iy = s' + iSs''$$

$$(3) \qquad \zeta = \xi + i\eta, \qquad |\zeta| \leqslant 1, \qquad 0 \leqslant \arg\zeta \leqslant \pi$$

$$(4) \qquad Z' = -\zeta^{-1} + C(W + \ln\zeta)$$

$$(5) \qquad Z'' = \ln\left[\frac{\zeta - \exp(i\alpha)}{\zeta - \exp(-i\alpha)}\right], \qquad \alpha < \arg Z'' \leqslant \pi + \alpha$$

where $S = \sin\alpha$, $C = \cos\alpha$, and values for α, U, V are as follows.

α	Range in U	Range in V	Number of Cases
0	$-2.8(.2)2$	$0(\pi/20)\pi$	525
$\pi/12$	$-2.4(.2)3.2$	$0(\pi/20)\pi$	609
$\pi/4$	$-2.4(.2)1.6$	$0(\pi/20)\pi$	441
$\pi/2$	$-2.4(.1)1.6$	$0(\pi/20)\pi/2$	451

x and y are given to 4D and ξ and η are given to 5D. In each case the error is less than a half a unit in last place.

[1] G. BIRKHOFF, & E. ZARANTANELLO, *Jets, Wakes and Cavities*, soon to be published.

AUTOMATIC COMPUTING MACHINERY

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TECHNICAL DEVELOPMENTS

The Incorporation of Subroutines into a Complete Problem on the NBS Eastern Automatic Computer

The construction of a modest-scale automatically-sequenced electronic digital computer at the National Bureau of Standards is nearing completion. This computer, called the NBS Eastern Automatic Computer (SEAC), is expected to be a useful tool both for numerical computation and for research in numerical analysis. In order to facilitate the exploitation of the SEAC, the Machine Development Laboratory of the Bureau has planned in advance key instruction routines for use on the machine. Groups of these instruction routines will be incorporated as subroutines in instruction programs governing the solution of complete problems. It is the purpose of this paper to show how subroutines prepared in advance of problem solution, selected from a library of permanent subroutines, will be properly inserted by the computer in the instruction program relevant to a problem at hand.

In order to follow this program it will be necessary to understand the principal features of the logical design of the SEAC.

1. Memory.—The present routine is written to fit the initial model of the machine which has 64 acoustic lines, or tanks, each line storing eight "words." A word consists of 48 binary digit positions, of which 45 are used to represent either a number or a four-address command, and the remaining three provide spacing between words. A number, N, of 44 binary digits followed by a sign digit, is stored as an absolute value with the proper sign attached. A plus sign is represented by "0" and a minus sign by "1." The binary point of the number is located between the second and third positions from the left so that $|N| < 4$.

In the operation representation, 10 binary positions are apportioned to each of four addresses, or memory locations. As there are only 512 memory locations in the initial model, nine binary digits are sufficient to specify any address (six digits to indicate the tank number and three digits to indicate the word within that tank). The first binary digit from the left of any address will always be zero and therefore will not be indicated here. The remaining nine binary digits will be represented by three octal digits. In addition, four binary digits represent, in coded form, the operations to be performed, designated herein by capital letters. The 45th digit from the left is again a sign indicator.

The following important feature of the acoustic-line memory should be kept in mind: information sent to an address will replace the previous content of that address.

2. Command Code.—In Table I under the headings of each of the four addresses α, β, γ, and δ the items which are located therein are listed. An address enclosed in parentheses indicates the content of the corresponding memory location; the address of the next command is abbreviated as "n.c."

The 45th binary digit from the left in a command is normally a zero (i.e., a plus sign); when a minus sign is coded in that position, the machine

TABLE I

Operation	α	β	γ	δ	Operation Symbol	Result of Operation
Addition	Augend	Addend	Sum	n.c.	A	$(\gamma) = (\alpha) + (\beta)$
Subtraction	Minuend	Subtrahend	Difference	n.c.	S	$(\gamma) = (\alpha) - (\beta)$
Multiplication (High-order)	Multiplicand	Multiplier	High-order Product	n.c.	M	(γ) = the more significant half of $(\alpha)(\beta)$
Multiplication (Low-order)	Multiplicand	Multiplier	Low-order Product	n.c.	N	(γ) = the less significant half of $(\alpha)(\beta)$
Multiplication (Rounded)	Multiplicand	Multiplier	Rounded Product	n.c.	R	(γ) = the more significant half of $(\alpha)(\beta)$ increased by the first digit of the less significant half of the product
Division	Divisor	Dividend	Quotient	n.c.	D	$(\gamma) = (\beta)/(\alpha)$, unrounded
Logical Transfer	Extractee	Extractor	Altered Number	n.c.	L	Digits in (γ) which correspond to 1's in (β) are replaced by corresponding digits of (α)
Algebraic Comparison	1st Comparand	2nd Comparand	n.c. or	n.c.	C	(γ) = n.c., if $(\alpha) < (\beta)$ (δ) = n.c., if $(\alpha) \geq (\beta)$
Absolute Value Comparison	1st Comparand	2nd Comparand	n.c. or	n.c.	K	(γ) = n.c., if $\lvert(\alpha)\rvert < \lvert(\beta)\rvert$ (δ) = n.c., if $\lvert(\alpha)\rvert \geq \lvert(\beta)\rvert$
Input Order	Any Number	Odd Number	Word from Input	n.c.	T	(γ) = next word from Input medium
Input Order	Any Number	Even Number	Word from Input	n.c.	T	(γ) to $(\gamma + 7)$ = next 8 words from Input medium
Output Order	Any Number	Odd Number	Word to be Recorded	n.c.	P	(γ) = next word on Output medium
Output Order	Any Number	Even Number	Word to be Recorded	n.c.	P	(γ) to $(\gamma + 7)$ = next 8 words on Output medium

will stop automatically after executing the indicated operation. Commands can be manipulated in the arithmetic unit of the machine, since, to this unit, they are indistinguishable from numbers.

3. Timing.—A minor cycle, the time needed for one word to pass a given point in the machine, is equal to 48 microseconds. The time consumed for a given operation, located at address ϵ, can be obtained from the following formulae (where an underscored symbol represents the last octal digit of the indicated address):

1. For operations A, S, and L, $t = (\underline{\epsilon} - \underline{\beta}) + (\underline{\beta} - \underline{\alpha}) + (\underline{\alpha} - \underline{\gamma}) + (\underline{\gamma} - \underline{\delta})$, where 8 minor cycles must be added to each difference which is algebraically less than $+1$.
2. For operations M, N, R, and D, the above sum must be increased by 40 minor cycles if $(\underline{\alpha} - \underline{\gamma}) \geq 5$; otherwise, the sum must be increased by 48 minor cycles.
3. For operations C and K, two timings are possibie depending on the result of the comparison; the number of minor cycles is given by $(\underline{\epsilon} - \underline{\beta}) + (\underline{\beta} - \underline{\alpha}) + (\underline{\alpha} - \underline{\gamma})$, if $(\alpha) < (\beta)$, or $(\underline{\epsilon} - \underline{\beta}) + (\underline{\beta} - \underline{\alpha}) + (\underline{\alpha} - \underline{\delta})$, if $(\alpha) \geq (\beta)$. In this case, 8 minor cycles must be added to the first two differences whenever their value is less than $+1$ and to the third difference whenever its value is less than $+2$.

The following table indicates the range of execution time for each command:

Operation	Minimum Time minor cycles	Minimum Time microsec.	Maximum Time minor cycles	Maximum Time microsec.
C, K	4	192	25	1200
A, S, L	4	192	32	1536
M, N, R, D	48	2304	76	3648
T, P	About 2 seconds per word, at present			

4. Modification of Subroutines.—In 1947, SAMUEL LUBKIN considered the problem of modifying subroutines which were to be used in a given problem to fit the available machine storage. He proposed the use of a Base Number Command to simplify the task of adapting any subroutine to its position in the memory. GOLDSTINE & VON NEUMANN[1] have reported on the programming of such adaptation using the codes of the Institute for Advanced Study machine. A similar routine prepared for use on the SEAC is presented herein. Because of the restricted storage capacity of this machine, an effort has been made to pack frequently-used routines in the internal memory as economically as possible. The present routine, in which the assumption is made that n subroutines are to be incorporated within the given program, occupies only $17 + n$ cells. This number does not include, however, four temporary storage cells and the storage for nine constants which belong to a common pool (as explained in the next section).

A general explanation of the method used in the modification of subroutines for insertion in main instruction routines will perhaps be helpful to the reader. Each permanent subroutine is coded as though its first word were located at the address 0 100 000 000 (i.e., the binary equivalent of the octal address 400). All addresses in the subroutine which require modification, as introduced into the computer, are greater than 400. Thus, there will be a "1" present in the second position of every address which must be modified and a zero in the corresponding position of all other addresses which are less than 400.

In the present program, each of the required group of subroutines is inserted into the memory in the final locations it is to occupy. The modifying routine begins at position 050. All addresses of 400 or above contained within the subroutine are modified to fit the actual location of the subroutine in the memory. In substance, the four digits corresponding to the second digit from the left of each address within such words of the subroutine as need modification are extracted. The resulting number is multiplied by d, which represents the difference between the address occupied by the first word of the subroutine under consideration and the number 400 at which address this word was originally coded. The resulting product is added to the original word, and the addresses contained therein are thereby properly modified.

Consider a specific example illustrating this procedure. As stated above the addresses will be written in octal form, although they are stored in binary form. Suppose the following subroutine operation, originally coded at address 400, is placed in the memory at address 070:

Cell No.	α	β	γ	δ	Operation
070	401	046	401	415	A

In order to modify (070), subtract 400 from 070 octally with the result, $d = -310$. Next, extract the second binary digit from the left of each address giving a "1" in the case of α, γ, and δ and a "0" in the case of β. Multiply each extracted digit by $d = -310$, giving -310 in the α, γ, and δ positions of a memory cell and 000 in the β position, and add the result octally to (070). The modified operation is as follows:

Cell No.	α	β	γ	δ	Operation
070	071	046	071	105	A

After all the subroutines in a program are modified, the memory cells occupied by the modifying routine are available for other uses.

5. Conventions Used in Coding.—Before actually presenting the program, it will be necessary to explain some of the conventions used in the coding of this routine. An address enclosed within brackets is used to indicate the fact that the content of that storage location will vary. Braces are used in the ordinary algebraic sense.

The first two memory tanks (i.e., memory locations 000 through 017) are retained for temporary storage in the modification routine; cell 007 contains the instruction to be executed immediately after the routine is completed. Positions 020 through 047 serve as a common pool of frequently-used constants, which all subroutines employ. Of this pool, the present routine makes use of the following constants:

Memory Cell	Content
020	+ Zero
023	400 (in binary form) occupying the α space, zeros occupying the remaining binary positions
027	the number 2^{-8}, representing a unit in the last position of α
031	the number 2^{-30}
034	the number 2^{-28}, representing a unit in the last position of γ
041	the number 2^{-10}
042	$2^{3} - 2^{-8}$, representing 10 units occupying the α space
044	$2^{-18} - 2^{-28}$, representing 10 units occupying the γ space
047	Zero (at the start of the routine only)

The following notations will be used in the program:

Notation	Significance
a_{ij}	the address at which the j-th word of the i-th subroutine is located, where $i = 1, 2, 3, \cdots n$ and $j = 1, 2, 3, \cdots b_i$
b_i	the number of words to be modified in the i-th subroutine
$\overline{w}_{ij}$	the content of cell a_{ij} before modification
w_{ij}	the content of cell a_{ij} after modification
s_k	the binary digit of $\overline{w}_{ij}$ in position k from the left, where $k = 1, 2, 3, \cdots 45$

PROGRAM

Cell No.	α	β	γ	δ	Operation	No. of minor cycles	Result of Operation
050	070	044	055	062	L	14	(055) = 010 013 a_{i1} 061 A
062	070	031	012	063	N	55	(012) = $2^{-18}a_{i1}$ (see footnote 2)
063	041	012	012	067	D	52	(012) = $2^{-8}a_{i1}$
067	012	042	051	064	L	19	(051) = a_{i1} 020 010 066 A
064	012	023	012	051	S	11	(012) = $\{a_{i1} - 400\}2^{-8} = d$
051	[a_{ij}]	020	010	066	A	12	(010) = $\overline{w}_{ij}$

Cell No.	α	β	γ	δ	Operation	No. of minor cycles	Result of Operation
066	010	065	047	054	L	10	$(047) = s_2 2^0 + s_{12}2^{-10} + s_{22}2^{-20} + s_{32}2^{-30} = t$
054	047	012	013	055	M	63	$(013) = td = c$, correction
055	010	013	$[a_{ij}]$	061	A	14	$(a_{ij}) = \bar{w}_{ij} + c = w_{ij}$
061	051	070	052 / 056		K	15/11	Test as to whether b_i modifications have been made.
052	027	051	051	060	A	10	If b_i modifications have not been made, (051) and (055) are stepped up.
060	034	055	055	051	A	15	
056	[071]	020	070	057	A	16	All b_i modifications have been made; $(070) = [\{a_{i1} + b_i - 1\}2^{-8} + a_{i1}2^{-28}]$, where $i = 2, 3, \cdots, (n + 1)$.
057	027	056	056	053	A	12	(056) is stepped up.
053	020	070	050 / 007		K	19/20	Test as to whether n subroutines have been modified.

Temporary and Constant Storage

065	$2^0 + 2^{-10} + 2^{-20} + 2^{-30}$
070	$\{a_{11} + b_1 - 1\}2^{-8} + a_{11}2^{-28}$; $[\{a_{i1} + b_i - 1\}2^{-8} + a_{i1}2^{-28}]$
071	$\{a_{21} + b_2 - 1\}2^{-8} + a_{21}2^{-28}$
	. . . .
$067 + n$	$\{a_{n1} + b_n - 1\}2^{-8} + a_{n1}2^{-28}$
$070 + n$	Zero

In this program the commands are arranged in logical sequence. On the input medium, they would be arranged in numerical order of addresses of the memory cells containing them. The time consumed for each instruction to be modified is 6.5 milliseconds.

MDL Staff

NBSMDL

[1] H. H. Goldstine, J. von Neumann, *Planning and Coding of Problems for an Electronic Computing Instrument*, Part II, v. 3, Institute for Advanced Study, Princeton, N. J., 1948 [*MTAC*, v. 3, p. 541–542].

[2] Because the first two digits from the left in the result of low order multiplication on the SEAC are always zero, two instructions are required to put a_{i1} into the α position of memory cell 012.

Discussions

A Note on "Is" and "Might Be" in Computers

Recent press reports have aired the disagreement between the "brain" and the "antibrain" factions in the computer fraternity. The term "brain" is a bit fanciful and perhaps smells slightly like commercial advertising; however, I feel that some of the stories given to the press have been more definitely misleading. I would like to enter a plea for careful distinction between facts and fancies by scientific people who write and speak for the general public.

A specialist in one field must be particularly careful in talking to the public about matters not in his own field of specialization. A careless or exaggerated remark made to experts in a field will do little harm because

the audience will make suitable allowance for "embroidery," but a similar remark made to the general public may be taken seriously and may lead to real misunderstanding.

At the present critical period in the development of science, when scientists are being forced somewhat reluctantly to take a hand in international affairs, there is much loose talk on the part of those who were formerly in undisputed control. Instead of excusing similar inaccuracies on the part of scientists, this seems to me to emphasize our responsibility to the public to present fair, accurate, and dependable statements.

This note is frankly an appeal to those who issue frequent statements to the public press, to distinguish carefully between imagination and fact—between what "might be" and what "is." I fear that these categories have been confused on several occasions.

The analogy between computers and the brain or nervous system, for example, has been a useful aid to invention and, when properly qualified, to an understanding of the workings of a computer. Invention, as I imagine it, consists of selecting out of many random combinations of ideas some that serve a useful purpose. Any device, no matter how devoid of logic, that helps to form useful combinations should be accepted thankfully. In this role, by suggesting what might be, the "brain" analogy had been useful to me and probably to others, particularly in the formative period of computer development. Once a combination of ideas has been formed, however, it must stand on its own logical feet; one must then be careful not to distort facts to fit them into the analogy no matter how striking.

According to an article in the *Saturday Evening Post*, Feb. 18, 1950, computers are subject to psychopathic states which the engineer in charge cures by a "shock treatment" consisting of the application of excessively large voltages. I would be interested to know whether this is a "might be" or an "is." So far, I have been unable to find anyone who recommends quite so slap-dash a method of trouble-shooting. It would be too much like dropping a valuable watch on the floor to improve its time-keeping ability; granted that this treatment would work once in a while, it hardly seems the economical procedure in the long run.

Another "might be" is the checking method that the reader of the public press would infer is generally used, with three parallel computing elements, all doing the same problem and voting on the correct answer. A majority vote decides the answer that the computer will print. There may be such a machine, despite the low economy of this checking method, but to date I have been unable to find this remarkably democratic device.

The listing of "might be's" could be extended to greater length. I should like to think, however, that I have touched upon a matter of concern to the entire scientific fraternity and that it is unnecessary to belabor my point. In all sincerity, I am urging more care and restraint in reporting scientific work to the public, particularly when the person reporting has only a secondary interest in that work and is likely to be misled by incomplete information.

G. R. STIBITZ

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Bibliography Z–XII

1. E. G. Andrews & H. W. Bode, "Use of relay digital computer," *Electrical Engineering*, v. 69, 1950, p. 158–163.

Use of two Bell Laboratories large-scale digital computers is described; discussion of process of setting up problem, decisions which must be made by attending mathematician, and those decisions which can be left to machines are noted; samples of different types of problems which have been solved successfully by machines are given; and procedures used are discussed.

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NBSMDL

2. Anon., *Description and Use of the ENIAC Converter Code*, Ballistic Research Laboratories, *Technical Note* No. 141, Aberdeen Proving Ground, Maryland, November 1949, 23 pages. Mimeographed.

In the first three years of its existence, the ENIAC, in spite of its frequent and often prolonged breakdowns, executed its tasks so speedily that the attending crew of programmers found it exceedingly hard to keep pace with its operation. The introduction of a set of coded orders by John von Neumann and Aberdeen's mathematical staff materially lessened the effort and time consumed by the attendants in supplying problems for the ENIAC. A new unit, the Converter, was incorporated which receives the code digits and initiates the corresponding operations. The machine operates on a one-address control system.

The list of orders falls into essentially three categories. The storage orders call for the machine to perform the following operations:

a) Transmit information from an IBM card to the eight Transmitter Groups, each holding a signed 10-decimal-digit number.
b) Transmit information from the above eight groups and from an additional pair of similar nature to 20 Registers (18 of which are misnamed Accumulators, as pointed out by the authors themselves), each holding a signed 10-decimal-digit number.
c) Transmit information from the three Function Tables to the above Registers. (A Function Table consists of 104 lines, each holding two signed six-digit numbers or a sequence of coded instructions. Each line bears a three-digit address.)

The arithmetic orders call for the machine to perform the following operations:

a) Add a number to the contents of an Accumulator.
b) Subtract a number from zero.
c) Multiply two numbers with single precision, and add the product to a third number.
d) Divide two numbers with single precision, with remainder available.
e) Extract square root, with remainder available.
f) Obtain the absolute value of a number.
g) Obtain $(1 - N)10^k$, where N is a number whose decimal point is k places from the left.
h) Multiply a number by 10^p, where $-5 \leq p \leq 5$. (These shift orders are of two types, allowing for the retention of the shifted-out digits in another Register, or for their deletion.)

The control orders call for the machine to perform the following operations:

a) Transfer Control—both conditionally and unconditionally.
b) Count in order to facilitate iterations.
c) Delay and stop.

The publication gives a careful explanation of each order but unfortunately exhibits some of the ills always accompanying the appending of an "afterthought." For instance, the letters A and B are used to indicate the first two Transmitter Groups, as well as the two sides of each Function Table line. The letter X, often misprinted as x, represents the Multiplication Code as well as an arbitrary number within a register. A slightly more serious error is the wrong illustration of order 6L, at the top of page 10, which accompanies the correct verbal explanation.

The interested reader would like to know whether there is any provision in the ENIAC for detecting overflow on addition and whether the multiplication order gives a rounded product. This technical note was designed for use within the Ballistic Research Laboratories and was written primarily for issuing available information when the occasion demands speed. The exposition of the codes is valuable in that it is sufficiently clear to enable any reader to attempt programming computations on the ENIAC.

IDA RHODES

NBSMDL

3. ANON., *Digital Computer Research at Birkbeck College*, Office of Naval Research (London Branch), *Technical Report* OANAR-50-49, 12 December 1949, 2 p.

This report describes the design features and operating characteristics of the three digital computers being constructed under the direction of A. D. BOOTH of Birkbeck College, University of London. They are the automatic relay computer (ARC), the simple electronic computer (SEC), and the all-purpose electronic x-ray computer (APEXC).

The ARC is a part relay, part electronic parallel machine using magnetic drum storage. Teletype tape is used as an input medium, and data is entered into the machine in binary form. A teleprinter or tape perforator will be used for output. Several simple problems have been run on this machine such as tabulation of prime numbers and factorization of large numbers. Although it is now temporarily dismantled, it will be reassembled and will use cross-bar type memory units.

The SEC is a small, all-electronic, parallel calculator. It has the same type of magnetic drum memory as the ARC; teletype is used for input and output.

The APEXC will be an evolution of the SEC using a magnetic drum store and employing about 800 vacuum tubes. The machine will store 1,024 numbers of 32 binary digits. Input and output will be on magnetic tape. The machine will be used initially for making crystallographic computations.

EDITH T. NORRIS

NBSMDL

4. ANON., *The EDSAC Computing Machine, Cambridge University*, Office of Naval Research (London Branch), *Technical Report* OANAR-43-49, 25 November 1949, 5 p.

This report describes the design features and present use of the EDSAC built by M. V. WILKES and his co-workers at the University Mathematical Laboratory, Cambridge, England. [See *MTAC*, v. 4, p. 61–65.] In

order to gain experience with the programming of problems on the EDSAC, a number of simple computational problems have been run on the machine (e.g., the calculation of prime numbers, tabulation of the complex gamma function [UMT 102], computation of the number "e" to 200 decimal places, and calculation of the AIRY integrals).

EDITH T. NORRIS

NBSMDL

5. WARREN S. McCULLOCH & JOHN PFEIFFER, "Of digital computers called brains," *The Scientific Monthly*, v. 69, 1949, p. 368–376.

Comparison between known operations in the human brain and similar operations in large-scale computers.

6. H. J. McSKIMIN, "Theoretical analysis of the mercury delay line," Acoustical Society of America, *Jn.*, v. 20, 1948, p. 418–424, 4 figs., 2 tables.

The electromechanical aspects of problems presented by the use of mercury delay lines in computers are discussed. After a brief discourse on the theory of wave propagation for liquids, an analysis is made of the voltage developed by the piezoelectric pick-up crystal used in the delay line and of the distortion that might result when the carrier pulse is modulated. It is found that waves with slightly differing phase velocities can exist and may produce distortion effects.

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NBSMDL

7. R. D. RICHTMYER & N. C. METROPOLIS, "Modern computing," *Physics Today*, v. 2, Oct. 1949, p. 8–15.

Developments during the last decade of automatic computing methods with rapid progress involving engineering principles and techniques, mathematical methods, and logic of automatic computation are discussed. Also described are the functions of the ENIAC, methods of presenting problems to computing machines, possibilities of using machines for analytic purposes, and practical uses of computers in their present stage of development.

EDITH T. NORRIS

NBSMDL

8. A. E. SMITH & C. V. L. SMITH, "Digital computers and their applications," American Society of Naval Engineers, *Jn.*, v. 61, 1949, p. 137–168, diags.

Presented herein is a general discussion of the design principles of digital computers and a description of existing machines and those under development.

NEWS

Association for Computing Machinery.—Rutgers University was host to members of the Association at a conference on automatic computing machinery. The conference included general sessions on Tuesday morning and evening, March 28, plus two parallel groups of sectional meetings designated as Section A and Section B, March 28 and 29.

In Section A, the talks dealt with applications of computers, engineering descriptions of several complete machines, and engineering problems encountered in the design of machine components. Section B was concerned with the mathematical aspects of the design of automatic computers, the coding of complete problems to be solved on the machines, and mathematical descriptions of complete machines.

In addition to the program of talks, the following exhibits were shown: the REAC by the Reeves Instrument Corporation, the RCA Linear Simultaneous Equation Solver by the RCA Laboratories, the IBM Card-Programmed Electronic Calculator by the International Business Machines Corporation, the small Binary-Octal Calculator by the Raytheon Manufacturing Company, the MADDIDA (Magnetic Drum Digital Differential Analyzer) by Northrop Aircraft, Incorporated, the oscillograph in the computer field by Allen B. DuMont Company, and Zator card equipment by the Zator Company.

The program was as follows:

General Session, Tuesday, March 28, Dean ELMER C. EASTON, College of Engineering, Rutgers Univ., presiding:

Welcoming address by MASON W. GROSS, Provost, Rutgers University.

"High speed computing machines—a survey," PERRY CRAWFORD, Research and Development Board.

"Machines of moderate cost," GEORGE STIBITZ, Burlington, Vt.

Section A, EDWARD W. CANNON, National Bureau of Standards, presiding.

"Applications of computing machines to the solution of management problems," MARSHALL K. WOOD, Department of the Air Force.

"Engineering applications of electronic analog computers," HERBERT ZAGOR, Reeves Instrument Corporation.

"Applications of electronic computers to Census Bureau problems," JAMES L. MCPHERSON, FLORENCE K. KOONS, RALPH E. MULLENDORE, Bureau of Census.

"Applications of the BINAC," JOHN W. MAUCHLY, Eckert-Mauchly Computer Corporation.

Section B, FRED G. FENDER, Rutgers University, presiding.

"The BINAC—a technical report," JAMES R. WEINER, Eckert-Mauchly Computer Corporation.

"An analog series computer," MAX G. SCHERBERG, Office of Air Research.

"Preliminary design of the Mark IV," BENJAMIN L. MOORE, Harvard Univ.

"Design of a low-cost computer" (read by C. V. L. SMITH), PAUL L. MORTON, Univ. of Calif.

General Session, banquet, HOUSTON PETERSON, Rutgers Univ., toastmaster.

"Automatic computing machinery of moderate cost," HOWARD H. AIKEN, Director, Computation Laboratory, Harvard Univ.

Section A, Wednesday, March 29, JAMES J. SLADE, Jr., Rutgers Univ., presiding.

"A digital computer for solution of simultaneous linear equations," SAMUEL LUBKIN, Electronic Computer Corporation.

"The ANACON, a large-scale general-purpose analog computer," D. L. WHITEHEAD, Westinghouse Electric Corporation.

"The IBM card-programmed electronic calculator," CUTHBERT C. HURD, International Business Machines Corporation.

"The MADDIDA, general features," FLOYD G. STEELE, Northrop Aircraft, Inc.

Section B, FRANZ L. ALT, National Bureau of Standards, presiding.

"Planning and error consideration in the numerical integration of a difference equation," FRANCIS J. MURRAY, Columbia Univ.

"Probability methods in the solution of elliptic partial differential equations," JOHN H. CURTISS, National Bureau of Standards.

"A machine method for solution of systems of ordinary differential equations," RICHARD F. CLIPPINGER & BERNARD DIMSDALE, Aberdeen Proving Ground.

"The theory of digital handling of nonnumerical information and its implications to machine economics," CALVIN M. MOOERS, Zator Company.

Section A, ERNEST G. ANDREWS, Bell Telephone Laboratories, presiding.

"New circuits installed in the Aiken relay calculator," FREDERICK G. MILLER, U. S. Naval Proving Ground.

"Digital computing machine components of universal application," WILLIAM S. ELLIOTT, Research Laboratories of Elliott Brothers, Ltd. (London).

"Design features of a magnetic drum information storage system," JOHN L. HILL, Engineering Research Associates, Inc.

"The SB-256 electrostatic selective storage tube," JAN A. RAJCHMAN, RCA Laboratories.

Section B, ELLIS R. OTT, Rutgers Univ., presiding

"Optical ray tracing," DONALD P. FEDER & BENJAMIN HANDY, National Bureau of Standards.

"Solution of matrix equations of high order by an automatic computer," HERBERT F. MITCHELL, Jr., Eckert-Mauchly Computer Corporation.

"Theodolite reductions on the IBM relay calculators," MARK LOTKIN & C. E. JOHNSON, Aberdeen Proving Ground.

"The MADDIDA, design features," DONALD E. ECKDAHL, Northrop Aircraft, Inc.

The Institute of Radio Engineers.—At the 1950 National Convention held in New York City from March 6 through March 8, two sessions were devoted to high-speed computers.

The first session on Wednesday morning, March 8, under the chairmanship of GEORGE R. STIBITZ, dealt exclusively with digital computers. In a paper entitled, "Static magnetic pulse control and information storage," AN WANG of Harvard University mentioned the possibility of developing a digital memory system using magnetic materials having rectangular hysteresis loops to control transfer of electrical pulses through the core by means of the property of residual magnetism. The incorporation of unidirectional current devices permits a static, high-efficiency pulse power distribution system. RALPH SLUTZ of the National Bureau of Standards next discussed the development of a generalized procedure of designing diode circuits for pulse gating, with particular attention to the extremely high reliability required for electronic digital computers. In the third paper, "Marginal checking as an aid to computer reliability," NORMAN H. TAYLOR of M.I.T. pointed out the source of error in digital computing and pulse communication caused by deterioration of machine components. Marginal checking varies voltages in logical circuit groups, inducing inferior parts to cause failure, while a test program or pulse transmission detects and localizes potential failure. By automatically carrying out this marginal checking, a digital computer can act as its own detector. "The development of the California Digital Computer" by DAVID R. BROWN and PAUL L. MORTON, University of California, Berkeley, presented a description of a low-cost general-purpose electronic digital computer having a compact magnetic-drum memory with a capacity for 10,000 ten-decimal-digit numbers. This large memory capacity permits much computation without the necessity of automatic access to input. The average access time is eight milliseconds. Machine input is accomplished on punched paper tape read photoelectrically. Binary-coded decimal numbers pass serially through the arithmetic unit on four parallel channels, and single-address operations (including division and square rooting) will be performed at the rate of about 60 per second. In the final talk of this session JOSEF KATZ, University of Toronto, mentioned a new class of switching tubes designed to reduce the complexity of electronic switching circuitry. The tube electrodes (one for each input and output channel) are designed so that particular combinations of input voltages result in current flow to the corresponding combination of output electrodes. The total current, and hence the cross section of the electron beam, is restricted by a limiting resistor used to focus the beam.

The second computer session on Wednesday afternoon, March 8, under the chairmanship of WILLIAM H. HUGGINS, was concerned with information analysis and computing. The REAC (Reeves Electronic Analog Computer) was described by H. I. ZAGOR, Reeves

Instrument Corporation, New York, as a flexible and practical tool for solving ordinary linear and nonlinear differential equations. In addition, the speaker illustrated the capabilities of the machine components and showed the various techniques involved in solving such representative problems as flutter, electron flow, automatic pilot design, Fourier analysis, engine control, integral and boundary value equations. "An electronic storage system" by E. W. BIVANS and J. V. HARRINGTON, Air Force Research Laboratories, Cambridge, Mass., described the RCA Radechon (a barrier grid storage tube) which is being used in a digital storage system. The secondary collector system is not used; instead, the reading signals are measured at the back plate, the same electrode on which the write signals are impressed. Deflection voltages are generated by a weighted addition of the plate voltages of a binary counter. Read and write operations are asynchronous with a 12-μ sec minimum time between operations. An application of the theory of correlation functions used in the determination of the transfer functions of linear and nonlinear systems was presented next by J. B. WIESNER and Y. W. LEE, M.I.T. In the paper, "Measurement and analysis of noise in a fire-control radar" by R. H. EISENGREIN, Sunstrand Machine Tool Company, Rockford, Illinois, an "instantaneous subtraction" method of optically measuring radar noise in order to analyze the unwanted portion of the signal return from an airborne target was discussed. Finally, H. E. SINGLETON, M.I.T., described a new electronic correlator which is capable of accepting inputs covering a wide frequency range and which evaluates correlation functions for arguments from 0 to 0.1 seconds. In order to obtain a high degree of accuracy and stability, the signals are sampled and converted to binary numbers, and the storage and computation are carried out digitally.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z-XII

9. D. P. ADAMS & H. T. EVANS, "Developments in the useful circular nomogram," *Rev. Sci. Instruments*, v. 20, 1949, p. 150–154.

A circular nomogram for $W = UV$ with the U, V scales on the circumference and the W scale on a diameter is described, with a discussion of the most advantageous choice of scales.

10. J. A. BRONZO & H. G. COHEN, "Note on analog computer design," *Rev. Sci. Instruments*, v. 20, 1949, p. 101–103.

This note proposes that partial differential equation problems be attacked by transforming them into difference-differential systems and then into systems of linear equations. The characteristic roots of the matrices of the latter are to be investigated by a change of coordinate system. The choice of the new coordinate system is not specified but is dependent on the ingenuity of the investigator.

F. J. M.

11. J. H. FELKER, "Calculator and chart for feedback problems," I.R.E. *Proc.*, v. 37, 1949, p. 1204–1206.

The chart consists essentially of constant-magnitude and constant-phase loci of z in the complex plane of $\gamma = z/(1 + z)$. A calculator based on this chart is available commercially.

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12. PHILLIP G. HUBBARD, "Application of electrical analogy in fluid mechanics research," *Rev. Sci. Instruments*, v. 20, 1949, p. 802–806.

Fluid flow problems are solved approximately by the measurements on the electric potential in a suitably shaped electrolytic tank. The electric potential is analogous to the velocity potential of the flow, inasmuch as both satisfy LAPLACE's equation and analogous boundary conditions can be obtained. Two techniques are described. In one of these, lucite models are immersed in a tank to simulate the flow pattern around the model and in the second, a tank segment with triangular cross sections was used to determine boundary pressures for a flow confined by various surfaces of revolution. A graphical comparison is made with experiment and the discrepancies analyzed.

F. J. M.

13. J. LYMAN & P. V. MARCHETTI, "A device for facilitating the computation of the first four moments about the mean," *Psychometrika*, v. 15, 1950, p. 49–55.

The device consists of a box with a number of endless tapes in it, arranged so that a chosen line on each tape can be read through a window in the box. A given distribution is divided into equally spaced intervals. Each tape corresponds to an interval x and each line corresponds to a possible frequency f for this interval. On this line f, fx, fx^2, fx^3, and fx^4 are typed and thus when each tape has been adjusted to the correct frequency for the given distribution, one can immediately read off the various terms which appear in expression for the first four moments.

F. J. M.

14. M. W. MAKOWSKI, "Slide rule for radiation calculations," *Rev. Sci. Instruments*, v. 20, 1949, p. 876–884.

This paper describes a special slide rule for making calculations connected with PLANCK'S radiation formula. By setting the cursor to a given temperature there may be found on the stock the total radiant energy and photon flux densities and the wave length of their maxima per unit wave length and per unit frequency as well as the values of their maxima. By setting the slide for a given temperature and the cursor for a given wave length, values of the ratios of the densities at any wave length to their maxima and of the integrated energy or number of quanta below or above that wave length may be read on the slide—WIEN'S law makes this possible. It is then simple to obtain actual densities and total fluxes within wave length ranges. Scales relating wave length, wave number, and electron volts are on the back of the stock. The accuracy is estimated by the inventor at from a few tenths of a per cent to five per cent, being of the order of one per cent in most ranges.

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15. R. E. MEYEROTT & G. BREIT, "Small differential analyzer with ball carriage integrators and selsyn coupling," *Rev. Sci. Instruments*, v. 20, 1949, p. 874–876.

This article describes a small differential analyzer built at Yale University around three ball carriage type mechanical integrators. The advantages of this type of integrator are that it has a good output torque, the input and output shafts are mechanically convenient, and the units are readily available as war surplus equipment. Coupling between integrators and input or output devices is by means of selsyns instead of shaft couplings, which makes the equipment very flexible and minimizes the set up time for new problems. The selsyns rotate at high speed (2500 to 1 to the integrator) to eliminate selsyn error. The analyzer also includes three input or output devices (called "function units") and an "adder" consisting of two selsyns coupled to a mechanical differential.

A test problem set in to generate a pure sine wave gave an increase of amplitude of 0.2 per cent per cycle, which compares favorably with results obtained with more elaborate machines. The device has proved satisfactory and convenient for the solution of problems with a nominal accuracy of one per cent.

SIDNEY GODET

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16. CYRIL STANLEY SMITH, "Blowing bubbles for a dynamic crystal model," *Jn. Applied Physics*, v. 20, 1949, p. 631.

This communication deals with the development of a certain type of analogue solution for problems in crystal structure and describes an improved method, using a nozzle, for obtaining the bubble rafts used by BRAGG and NYE for metal crystal study. (*Cf.* R. Soc. London, *Proc.*, v. 190 A, 1947, p. 474–481.)

F. J. M.

NOTES

116. ON FINDING THE SQUARE ROOT OF A COMPLEX NUMBER.—There are two classical methods used for finding x and y in

$$(a + bi)^{\frac{1}{2}} = x + yi. \tag{1}$$

The first of these is based on DEMOIVRE's theorem. One computes in succession

$$R = (a^2 + b^2)^{\frac{1}{2}} \qquad \phi = \arctan (b/a)$$
$$x = R \cos \tfrac{1}{2}(\phi + 2K\pi) \qquad y = R \sin \tfrac{1}{2}(\phi + 2K\pi), \qquad (K = 0, 1).$$

The second method is derived from purely algebraic considerations. From (1) it follows that

$$x^2 - y^2 = a, \qquad 2xy = b \tag{2}$$
$$R = x^2 + y^2 = (a^2 + b^2)^{\frac{1}{2}}$$
$$x = \pm\, ((R + a)/2)^{\frac{1}{2}}$$
$$y = \pm\, ((R - a)/2)^{\frac{1}{2}}. \tag{3}$$

The signs are determined in accordance with (2). The second method avoids the use of trigonometric lookups. In addition, the first method is reducible to the second. Hence, the second method is the better one to use. However, it, too, has some serious shortcomings. When b is small, either x or y suffers from a loss in significance. How does one prevent such a loss?

GARRETT BIRKHOFF has suggested a third procedure. Define

$$t = \tfrac{1}{2}(R + |a|). \tag{4}$$

Then x and y are determined as follows:

If $a > 0$, $x = t^{\frac{1}{2}}$, $y = \frac{1}{2}bt^{-\frac{1}{2}}$.
If $a < 0$, $x = \frac{1}{2}bt^{-\frac{1}{2}}$, $y = t^{\frac{1}{2}}$.
If $a = 0$, $x = y = (b/2)^{\frac{1}{2}}$.

Of course in all cases, the signs of x and y are determined in accordance with (2).

The numerical analyst should prefer the third method for two reasons: (a) accuracy is preserved, (b) the square root in (3) is replaced by a division, a great time-saving advantage.

Example: Find the square root of $-7 + .03i$.

The second method gives only

$$x = .00566\ 95 \qquad y = 2.64575\ 7386$$

while the third method gives

$$x = .00566\ 94540\ 7 \qquad y = 2.64575\ 7386.$$

Should any of the readers find an exposition of this technique, I should be glad to know about it. Birkhoff and I believe it to be new.

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117. TABLE FOR SOLVING CUBIC EQUATIONS.—Judging by the title of the following 75-page pamphlet published more than sixty years ago one would never suspect that in it might be any table of mathematical interest: JOHN BORDEN, *Some Higher Plane Curves*, Chicago, E. T. Decker, 1888. Curves of the third, fourth, and fifth degrees are considered; but on pages 55–75 is a table of the roots of $m^3 - 3m + C = 0$ for values of C between $+2$ and -2. Each column headed respectively 1, 2, 3, contains the values of a root in numerical order. The value of C next to the right of the column of roots has its proper value for C directly opposite and to the right of it. In the first column headed 3, the values are successively $\sqrt{3}$, 1.733(.001)2, and corresponding to each of these values is a 9D value for C. In the column headed 1, the values range 0(.001)1; in the column headed 2, the range is $\sqrt{3}$, 1.731(−.001)1. The author remarks: "The value of each root has its proper value for C directly opposite and to the right of it." "When the value of C is positive, roots nos. 1 and 2 are positive and no. 3 is negative: and when C is negative nos. 1 and 2 are negative and no. 3 is positive." Corresponding to the root 3 = 1.949, there are the corresponding root 1 = .945 and the root 2 = 1.055, followed by 54 other pairs .946(.001).999 and

1.054(–.001)1.001. To each of these 109 roots is given the corresponding value of C to 9D. Then comes the final entry in the table: root 3 = 2, $C = 2$; root 1 = 1, $C = 2$; root 2 = 1, $C = 2$.

R. C. A.

118. Note on the paper "On a definite integral" by R. H. Ritchie [*MTAC*, v. 4, p. 75–77].—The value given by Ritchie for the integral

$$f(x) = \int_0^\infty (u + x)^{-1} \exp(-u^2) du$$

is not free of ambiguity, since the definition of $\mathrm{Ei}(y)$ given in the paper does not make sense for $y = x^2 > 0$. The difficulty arises over the inversion of the Laplace transform (L.T.) $p^{-1} \ln (p - 1)$.

Moreover, $f(x)$ can be evaluated by a routine process, since it is simply[1] the repeated L.T. of $\exp(-u^2)$. The L.T. of $\exp(-u^2)$ is[2]

$$\tfrac{1}{2}\pi^{\frac{1}{2}} \exp(\tfrac{1}{4}t^2) \operatorname{erfc}(\tfrac{1}{2}t)$$

where

$$\operatorname{erfc} z = 2\pi^{-\frac{1}{2}} \int_z^\infty \exp(-u^2) du.$$

Next, the L.T. of this can be obtained by means of Campbell & Foster,[2] formula 959.5. The result is incorrectly rendered in Cossar & Erdélyi;[3] the corrected form is

$$-\tfrac{1}{2} \exp(-x^2)\mathrm{Ei}(-x^2e^{i\pi}) + \tfrac{1}{2}i\pi \exp(-x^2) \operatorname{erfc}(xe^{i\pi/2}).$$

This can be expressed in terms of real valued functions if we use the function[4]

$$\overline{\mathrm{Ei}}(z) = \tfrac{1}{2}\mathrm{Ei}(-ze^{i\pi}) + \tfrac{1}{2}\mathrm{Ei}(-ze^{-i\pi}).$$

The result is

$$f(x) = \exp(-x^2)\left[\pi^{\frac{1}{2}} \int_0^x \exp(u^2)du - \tfrac{1}{2}\overline{\mathrm{Ei}}(x^2)\right].$$

A. E.

[1] D. V. Widder, *The Laplace Transform*, Princeton, 1941, Chapter VIII, 1.

[2] G. A. Campbell & R. M. Foster, *Fourier Integrals for Practical Applications*, New York, 1948, no. 903.0.

[3] J. Cossar & A. Erdélyi, *Dictionary of Laplace Transforms*, London, 1944–1946, p. VI, 76. In entry 3, read $(2\alpha i\pi^{\frac{1}{2}})^{-1}$ instead of $i\pi^{\frac{1}{2}}(2\alpha)^{-1}$; in entry 4 read erfc $[p/(2\alpha)]$ in place of erfc $(p\alpha^{\frac{1}{2}})$, and $\pi^{-1}\mathrm{Ei}$ in place of Ei.

[4] E. Jahnke & F. Emde, *Tables of Functions*, Leipzig, 1938, p. 2.

119. Harry Bateman Bibliography.—In *MTAC*, v. 3, p. 141–142 we presented a complete summary of published biographical sketches of Bateman, and lists of his publications, as well as a supplement to such lists of material published in the *Educational Times* (*E.T.*) and *Educational Times Reprint* (*E.T.R.*). Miss C. Brudno, in the Applied Mathematics Branch, Mechanics Division, of the Naval Research Laboratory, Washington, has noted the following five additions to this latter list of problems proposed: 14943, *E.T.*, v. 54, 1901, p. 328; sols. by Bateman and another, *E.T.R.*, v. 1, 1902, p. 98–100. 14975, *E.T.*, v. 54, 1901, p. 423; sols. *E.T.R.*, n.s. , v. 2, 1902, p. 111, v. 3, 1903, p. 29. 15119, *E.T.*, v. 55, 1902, p. 233; so l. v. 55, 1902,

p. 516, also *E.T.R.*, n.s., v. 3, 1903, p. 110–111. 15221, *E.T.*, v. 55, 1902, p. 438; sol. *E.T.R.*, n.s., v. 4, 1903, p. 88. 16009, *E.T.*, v. 59, 1906, p. 270; sol. *E.T.R.*, n.s., v. 11, 1907, p. 57–61.

R. C. A.

QUERY

35. W. THIELE'S TABLE.—According to KAYSER'S VOLLSTÄNDIGES BÜCHERLEXICON and HINRICH'S *Katalog*, *Tafel der Wolfram'schen hyperbolishen 48 stelligen Logarithmen. Bearbeitet und erweitert von W. Thiele*, was published in two printings, by different companies, 118 p., at Dessau: (1) 1905; (2) 1908. This publication appears to be extremely rare. There are copies of (2) in libraries of Harvard University, of Mr. C. R. COSENS, Cambridge, England, of Brown University (film), and in the John Crerar Library, Chicago. It was Cosens who, in 1939, directed my attention to (1). In what libraries is a copy of (1) located?

R. C. A.

QUERIES—REPLIES

45. THE INTEGRAL $\int_0^x e^{-A\sec\theta}d\theta$ (Q 19, v. 2, p. 196).—Attention is called to a manuscript table of this integral in UMT 103.

46. PITISCUS TABLES (Q 29, v. 3, p. 398; QR 40, p. 498–499, 42, p. 562–563).—For nearly 40 years I've had in my library BASIL ANDERSON & R. T. RICHARDSON, *Catalogue of the Books and Tracts on Pure Mathematics in the Central Library*, Newcastle-upon-Tyne, Newcastle-upon-Tyne, 1901. This library contains many old valuable books, but I did not earlier think to check for its possible Pitiscus items. On p. 33 it is indicated that the library owns copies of both 19966a (the *Canon*) and 19967 (the English *Trigonometry*, 1614), which therefore supplements the information we assembled [*MTAC*, v. 3, p. 499]. The library has also a copy of the 1612 German edition of the *Trigonometry* 6 [v. 3, p. 391].

R. C. A.

CORRIGENDA

V. 3, p. 54, l. 3, *for* 4–17, *read* 3–16.

V. 3, p. 406, l. 10 and 11, *interchange* $\cos \frac{1}{4}k\pi (k=1, 2, 3)$ and $\cos \frac{1}{8}(2k+1)\pi (k=0, 1, 2, 3)$.

V. 4, p. 21, l. 1 and 2, *for* Asymptotic distribution of range from that of reduced range *read* The distribution of the range. l. 7 *for* 193–196 *read* 395–396. l. 9 *delete* .1(.1).9(.01). l. –21, *for* 4 ∞ D *read* ∞.

V. 4, p. 23, l. –19, *for* 43–126 *read* 113–126.

V. 4, p. 29, l. –13, *for* p. xx *read* p. 11–15. l. –12, *for* Jardan *read* Jarden.

V. 4, p. 59, l. –1, *for* $a_{i1}a_{1j}/a_{11}$ *read* $a_{ij} - a_{i1}(a_{1j}/a_{11})$.

V. 4, p. 78, l. 25, *for* 1938 *read* 1935.

V. 4, p. 82, l. 14, *for* 548–553 *read* 948–953.

V. 4, p. 84, l. 5, *for* 5D *read* 4D.

V. 4, p. 91, l. 7, *for* U *read* u.

V. 4, p. 96, l.–23, *for* 37–41 *read* 33–41.

V. 4, p. 99, l.–23, *for* 332–371 *read* 322–371.

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